

ON OKOUNKOV'S CONJECTURE CONNECTING HILBERT SCHEMES OF POINTS AND MULTIPLE q -ZETA VALUES

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ABSTRACT. We compute the generating series for the intersection pairings between the total Chern classes of the tangent bundles of the Hilbert schemes of points on a smooth projective surface and the Chern characters of tautological bundles over these Hilbert schemes. Modulo the lower weight term, we verify Okounkov's conjecture [Oko] connecting these Hilbert schemes and multiple q -zeta values. In addition, this conjecture is completely proved when the surface is abelian. We also determine some universal constants in the sense of Boissière and Nieper-Wisskirchen [Boi, BN] regarding the total Chern classes of the tangent bundles of these Hilbert schemes. The main approach of this paper is to use the set-up of Carlsson and Okounkov outlined in [Car, CO] and the structure of the Chern character operators proved in [LQW2].

1. Introduction

In the region $\operatorname{Re} s > 1$, the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The integers $s > 1$ give rise to a sequence of special values of the Riemann zeta function. Multiple zeta values are series of the form

$$\zeta(s_1, \dots, s_k) = \sum_{n_1 > \dots > n_k} \frac{1}{n_1^{s_1} \dots n_k^{s_k}}$$

where n_1, \dots, n_k denote positive integers, and s_1, \dots, s_k are positive integers with $s_1 > 1$. Multiple q -zeta values are q -deformations of $\zeta(s_1, \dots, s_k)$, which may take different forms (see [Bra1, Bra2, OT, Zud] for details). In [Oko], Okounkov proposed several interesting conjectures regarding multiple q -zeta values and Hilbert schemes of points. Motivated by these conjectures, we compute in this paper the generating series for the intersection pairings between the total Chern classes of the tangent bundles of the Hilbert schemes of points on a smooth projective surface and the Chern characters of tautological bundles over these Hilbert schemes.

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Let X be a smooth projective complex surface, and let $X^{[n]}$ be the Hilbert scheme of n points in X . A line bundle L on X induces a tautological rank- n bundle $L^{[n]}$ on $X^{[n]}$. Let $\text{ch}_k(L^{[n]})$ be the k -th Chern character of $L^{[n]}$. Following Okounkov [Oko], we introduce the two generating series:

$$\langle \text{ch}_{k_1}^{L_1} \cdots \text{ch}_{k_N}^{L_N} \rangle = \sum_{n \geq 0} q^n \int_{X^{[n]}} \text{ch}_{k_1}(L_1^{[n]}) \cdots \text{ch}_{k_N}(L_N^{[n]}) \cdot c(T_{X^{[n]}}) \quad (1.1)$$

$$\langle \text{ch}_{k_1}^{L_1} \cdots \text{ch}_{k_N}^{L_N} \rangle' = \langle \text{ch}_{k_1}^{L_1} \cdots \text{ch}_{k_N}^{L_N} \rangle / \langle 1 \rangle = (q; q)_\infty^{\chi(X)} \cdot \langle \text{ch}_{k_1}^{L_1} \cdots \text{ch}_{k_N}^{L_N} \rangle \quad (1.2)$$

where $0 < q < 1$, $c(T_{X^{[n]}})$ is the total Chern class of the tangent bundle $T_{X^{[n]}}$, $\chi(X)$ is the Euler characteristics, and $(a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i)$. In [Car], for $X = \mathbb{C}^2$ with a suitable \mathbb{C}^* -action and $L = \mathcal{O}_X$, the series $\langle \text{ch}_{k_1}^L \cdots \text{ch}_{k_\ell}^L \rangle$ in the equivariant setting has been studied. In [Oko], Okounkov proposed the following conjecture.

Conjecture 1.1. $\langle \text{ch}_{k_1}^L \cdots \text{ch}_{k_N}^L \rangle'$ is a multiple q -zeta value of weight $\sum_{i=1}^N (k_i + 2)$.

In this paper, we study Conjecture 1.1. To state our result, we introduce some definitions. For integers $n_i > 0$, $w_i > 0$ and $p_i \geq 0$ with $1 \leq i \leq v$, define the *weight* of $\prod_{i=1}^v \frac{q^{n_i w_i p_i}}{(1 - q^{n_i})^{w_i}}$ to be $\sum_{i=1}^v w_i$. For $k \geq 0$ and $\alpha \in H^*(X)$, define $\Theta_k^\alpha(q, z)$ to be the weight- $(k + 2)$ multiple q -zeta value (with an additional variable z inserted):

$$\begin{aligned} & - \sum_{\substack{a, s_1, \dots, s_a, b, t_1, \dots, t_b \geq 1 \\ \sum_{i=1}^a s_i + \sum_{j=1}^b t_j = k+2}} \langle (1_X - K_X)^{\sum_{i=1}^a s_i}, \alpha \rangle \prod_{i=1}^a \frac{(-1)^{s_i}}{s_i!} \cdot \prod_{j=1}^b \frac{1}{t_j!} \\ & \cdot \sum_{n_1 > \dots > n_a} \prod_{i=1}^a \frac{(qz)^{n_i s_i}}{(1 - q^{n_i})^{s_i}} \cdot \sum_{m_1 > \dots > m_b} \prod_{j=1}^b \frac{z^{-m_j t_j}}{(1 - q^{m_j})^{t_j}} \end{aligned}$$

where K_X and 1_X are the canonical class and fundamental class of X respectively. Let $\text{Coeff}_{z_1^0 \dots z_N^0}(\cdot)$ denote the coefficient of $z_1^0 \cdots z_N^0$, L also denote the first Chern class of the line bundle L , and e_X be the Euler class of X .

Theorem 1.2. Let L_1, \dots, L_N be line bundles over X , and $k_1, \dots, k_N \geq 0$. Then,

$$\langle \text{ch}_{k_1}^{L_1} \cdots \text{ch}_{k_N}^{L_N} \rangle' = \text{Coeff}_{z_1^0 \dots z_N^0} \left(\prod_{i=1}^N \Theta_{k_i}^{1_X}(q, z_i) \right) + W, \quad (1.3)$$

and the lower weight term W is an infinite linear combination of the expressions:

$$\prod_{i=1}^u \left\langle K_X^{r_i} e_X^{r'_i}, L_1^{\ell_{i,1}} \cdots L_N^{\ell_{i,N}} \right\rangle \cdot \prod_{i=1}^v \frac{q^{n_i w_i p_i}}{(1 - q^{n_i})^{w_i}}$$

where $\sum_{i=1}^v w_i < \sum_{i=1}^N (k_i + 2)$, and the integers $u, v, r_i, r'_i, \ell_{i,j} \geq 0, n_i > 0, w_i > 0, p_i \in \{0, 1\}$ depend only on k_1, \dots, k_N . Furthermore, all the coefficients of this linear combination are independent of q, L_1, \dots, L_N and X .

Theorem 1.2 proves Conjecture 1.1, modulo the lower weight term W . Note that the leading term $\text{Coeff}_{z_1^0 \dots z_N^0} \left(\prod_{i=1}^N \Theta_{k_i}^{1_X}(q, z_i) \right)$ in $\langle \text{ch}_{k_1}^{L_1} \dots \text{ch}_{k_N}^{L_N} \rangle'$ has weight $\sum_{i=1}^N (k_i + 2)$, and is a multiple of $\langle K_X, K_X \rangle^N$ whose coefficient depends only on k_1, \dots, k_N and is independent of the line bundles L_1, \dots, L_N and the surface X .

In general, it is unclear how to organize the lower weight term W in Theorem 1.2 into multiple q -zeta values. On the other hand, we have the following result which together with Theorem 1.2 verifies Conjecture 1.1 when X is an abelian surface.

Theorem 1.3. *Let L_1, \dots, L_N be line bundles over an abelian surface X , and $k_1, \dots, k_N \geq 0$. Then, the lower weight term W in (1.3) is a linear combination of the coefficients of $z_1^0 \dots z_N^0$ in some multiple q -zeta values (with additional variables z_1, \dots, z_N inserted) of weights $< \sum_{i=1}^N (k_i + 2)$. Moreover, the coefficients in this linear combination are independent of q .*

We remark that some of the multiple q -zeta values mentioned in Theorem 1.3 are in the generalized sense, i.e., in the following form:

$$\sum_{n_1 > \dots > n_\ell} \prod_{i=1}^{\ell} \frac{(-n_i)^{w_i} q^{n_i p_i} f_i(z_1, \dots, z_N)^{n_i}}{(1 - q^{n_i})^{w_i}}$$

where $0 \leq p_i \leq w_i$, and each $f_i(z_1, \dots, z_N)$ is a monomial of $z_1^{\pm 1}, \dots, z_N^{\pm 1}$. We refer to (4.34) in the proof of Theorem 4.10 for more details. As indicated in [Oko], the factors $(-n_i)^{w_i}$ in the above expression may be related to the operator $q \frac{d}{dq}$.

The main idea in our proofs of Theorem 1.2 and Theorem 1.3 is to use the structure of the Chern character operators proved in [LQW2] and the set-up of Carlsson and Okounkov in [Car, CO]. Let $G_k(\alpha, n)$ be the degree- $(2k + |\alpha|)$ component of (2.1), and $\mathfrak{G}_k(\alpha)$ be the Chern character operator acting on the Fock space $\mathbb{H}_X = \bigoplus_{n=0}^{\infty} H^*(X^{[n]})$ via cup product by $\bigoplus_{n=0}^{\infty} G_k(\alpha, n)$. Then,

$$\text{ch}_k(L^{[n]}) = G_k(1_X, n) + G_{k-1}(L, n) + G_{k-2}(L^2/2, n)$$

by the Grothendieck-Riemann-Roch Theorem. So $\langle \text{ch}_{k_1}^{L_1} \dots \text{ch}_{k_N}^{L_N} \rangle'$ is reduced to the series $F_{k_1, \dots, k_N}^{\alpha_1, \dots, \alpha_N}(q)$ defined by (2.2). Let \mathfrak{L}_1 be the trivial line bundle on X with a scaling action of \mathbb{C}^* of character 1[†]. Using the set-up in [Car, CO], we get

$$F_{k_1, \dots, k_N}^{\alpha_1, \dots, \alpha_N}(q) = \text{Tr } q^{\mathfrak{d}} W(\mathfrak{L}_1, z) \prod_{i=1}^N \mathfrak{G}_{k_i}(\alpha_i)$$

where $W(\mathfrak{L}_1, z)$ is the vertex operator constructed in [Car, CO], and \mathfrak{d} is the number-of-points operator (i.e., $\mathfrak{d}|_{H^*(X^{[n]})} = n \text{Id}$). The structure of the Chern character operators $\mathfrak{G}_{k_i}(\alpha_i)$ is given by Theorem 2.3 which is proved in [LQW2].

[†]Throughout the paper, we implicitly set $t = 1$ for the generator t of the equivariant cohomology $H_{\mathbb{C}^*}^*(\text{pt})$ of a point.

It implies that the computation of $F_{k_1, \dots, k_N}^{\alpha_1, \dots, \alpha_N}(q)$ can be further reduced to

$$\mathrm{Tr} \, q^\partial W(\mathfrak{L}_1, z) \prod_{i=1}^N \frac{\mathfrak{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \quad (1.4)$$

where $\lambda^{(i)}$ denotes a *generalized* partition which may also contain negative parts, and $\lambda^{(i)}!$ and $\mathfrak{a}_{\lambda^{(i)}}(\alpha_i)$ are defined in Definition 2.2 (ii). The trace (1.4) is investigated via some standard but rather lengthy calculations.

As an application, our results enable us to determine some of the universal constants in $\sum c(T_{X^{[n]}}) q^n$. Let $C_i = \binom{2i}{i}/(i+1)$ be the Catalan number, and $\sigma_1(i) = \sum_{j|i} j$. By [Boi, BN], there exist unique rational numbers $b_\mu, f_\mu, g_\mu, h_\mu$ depending only on the (usual) partitions μ such that $\sum_n c(T_{X^{[n]}}) q^n$ is equal to

$$\exp \left(\sum_{\mu} q^{|\mu|} \left(b_{\mu} \mathfrak{a}_{-\mu}(1_X) + f_{\mu} \mathfrak{a}_{-\mu}(e_X) + g_{\mu} \mathfrak{a}_{-\mu}(K_X) + h_{\mu} \mathfrak{a}_{-\mu}(K_X^2) \right) \right) |0\rangle;$$

in addition, $b_{2i} = 0$, $b_{2i-1} = (-1)^{i-1} C_{i-1}/(2i-1)$, $b_{(1^i)} = f_{(1^i)} = -g_{(1^i)} = \sigma_1(i)/i$, and $h_{(1^i)} = 0$. In Theorem 6.4, we determine $b_{(i,1^j)}$ for $i \geq 2$ and $j \geq 0$.

The paper is organized as follows. In Sect. 2, we review the Heisenberg operators of Grojnowski and Nakajima, and the structure of the Chern character operators. In Sect. 3, we recall the vertex operator of Carlsson and Okounkov. In Sect. 4, we compute the trace (1.4). Theorem 1.2 and Theorem 1.3 are proved in Sect. 5. In Sect. 6, we determine the universal constants $b_{(i,1^j)}$ for $i \geq 2$ and $j \geq 0$.

Convention. All the (co)homology groups are in \mathbb{C} -coefficients unless otherwise specified. For $\alpha, \beta \in H^*(Y)$ where Y is a smooth projective variety, $\alpha\beta$ and $\alpha \cdot \beta$ denote the cup product $\alpha \cup \beta$, and $\langle \alpha, \beta \rangle$ denotes $\int_Y \alpha\beta$.

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2. Basics on Hilbert schemes of points on surfaces

In this section, we will review some basic aspects of the Hilbert schemes of points on surfaces. We will recall the definition of the Heisenberg operators of Grojnowski and Nakajima, and the structure of the Chern character operators.

Let X be a smooth projective complex surface, and $X^{[n]}$ be the Hilbert scheme of n points in X . An element in $X^{[n]}$ is represented by a length- n 0-dimensional closed subscheme ξ of X . For $\xi \in X^{[n]}$, let I_{ξ} be the corresponding sheaf of ideals. It is well known that $X^{[n]}$ is smooth. Define the universal codimension-2 subscheme:

$$\mathcal{Z}_n = \{(\xi, x) \in X^{[n]} \times X \mid x \in \mathrm{Supp}(\xi)\} \subset X^{[n]} \times X.$$

Denote by p_1 and p_2 the projections of $X^{[n]} \times X$ to $X^{[n]}$ and X respectively. Let

$$\mathbb{H}_X = \bigoplus_{n=0}^{\infty} H^*(X^{[n]})$$

be the direct sum of total cohomology groups of the Hilbert schemes $X^{[n]}$. For $m \geq 0$ and $n > 0$, let $Q^{[m,m]} = \emptyset$ and define $Q^{[m+n,m]}$ to be the closed subset:

$$\{(\xi, x, \eta) \in X^{[m+n]} \times X \times X^{[m]} \mid \xi \supset \eta \text{ and } \text{Supp}(I_\eta/I_\xi) = \{x\}\}.$$

We recall Nakajima's definition of the Heisenberg operators [Nak]. Let $\alpha \in H^*(X)$. Set $\mathbf{a}_0(\alpha) = 0$. For $n > 0$, the operator $\mathbf{a}_{-n}(\alpha) \in \text{End}(\mathbb{H}_X)$ is defined by

$$\mathbf{a}_{-n}(\alpha)(a) = \tilde{p}_{1*}([Q^{[m+n,m]}] \cdot \tilde{\rho}^* \alpha \cdot \tilde{p}_2^* a)$$

for $a \in H^*(X^{[m]})$, where $\tilde{p}_1, \tilde{\rho}, \tilde{p}_2$ are the projections of $X^{[m+n]} \times X \times X^{[m]}$ to $X^{[m+n]}, X, X^{[m]}$ respectively. Define $\mathbf{a}_n(\alpha) \in \text{End}(\mathbb{H}_X)$ to be $(-1)^n$ times the operator obtained from the definition of $\mathbf{a}_{-n}(\alpha)$ by switching the roles of \tilde{p}_1 and \tilde{p}_2 . We often refer to $\mathbf{a}_{-n}(\alpha)$ (resp. $\mathbf{a}_n(\alpha)$) as the *creation* (resp. *annihilation*) operator. The following is from [Nak, Gro]. Our convention of the sign follows [LQW2].

Theorem 2.1. *The operators $\mathbf{a}_n(\alpha)$ satisfy the commutation relation:*

$$[\mathbf{a}_m(\alpha), \mathbf{a}_n(\beta)] = -m \delta_{m,-n} \cdot \langle \alpha, \beta \rangle \cdot \text{Id}_{\mathbb{H}_X}.$$

The space \mathbb{H}_X is an irreducible module over the Heisenberg algebra generated by the operators $\mathbf{a}_n(\alpha)$ with a highest weight vector $|0\rangle = 1 \in H^0(X^{[0]}) \cong \mathbb{C}$.

The Lie bracket in the above theorem is understood in the super sense according to the parity of the cohomology degrees of the cohomology classes involved. It follows from Theorem 2.1 that the space \mathbb{H}_X is linearly spanned by all the Heisenberg monomials $\mathbf{a}_{n_1}(\alpha_1) \cdots \mathbf{a}_{n_k}(\alpha_k)|0\rangle$ where $k \geq 0$ and $n_1, \dots, n_k < 0$.

Definition 2.2. (i) Let $\alpha \in H^*(X)$ and $k \geq 1$. Define $\tau_{k*} : H^*(X) \rightarrow H^*(X^k)$ to be the linear map induced by the diagonal embedding $\tau_k : X \rightarrow X^k$, and

$$(\mathbf{a}_{m_1} \cdots \mathbf{a}_{m_k})(\alpha) = \mathbf{a}_{m_1} \cdots \mathbf{a}_{m_k}(\tau_{k*} \alpha) = \sum_j \mathbf{a}_{m_1}(\alpha_{j,1}) \cdots \mathbf{a}_{m_k}(\alpha_{j,k})$$

when $\tau_{k*} \alpha = \sum_j \alpha_{j,1} \otimes \cdots \otimes \alpha_{j,k}$ via the Künneth decomposition of $H^*(X^k)$.

(ii) Let $\lambda = (\cdots (-2)^{m_{-2}} (-1)^{m_{-1}} 1^{m_1} 2^{m_2} \cdots)$ be a *generalized partition* of the integer $n = \sum_i i m_i$ whose part $i \in \mathbb{Z}$ has multiplicity m_i . Define $\ell(\lambda) = \sum_i m_i$, $|\lambda| = \sum_i i m_i = n$, $s(\lambda) = \sum_i i^2 m_i$, $\lambda! = \prod_i m_i!$, and

$$\mathbf{a}_\lambda(\alpha) = \left(\prod_i (\mathbf{a}_i)^{m_i} \right) (\alpha)$$

where the product $\prod_i (\mathbf{a}_i)^{m_i}$ is understood to be $\cdots \mathbf{a}_{-2}^{m_{-2}} \mathbf{a}_{-1}^{m_{-1}} \mathbf{a}_1^{m_1} \mathbf{a}_2^{m_2} \cdots$.

The set of all generalized partitions is denoted by $\tilde{\mathcal{P}}$.

(iii) A generalized partition becomes a *partition* in the usual sense if the multiplicity $m_i = 0$ for all $i < 0$. The set of all partitions is denoted by \mathcal{P} .

For $n > 0$ and a homogeneous class $\alpha \in H^*(X)$, let $|\alpha| = s$ if $\alpha \in H^s(X)$, and let $G_k(\alpha, n)$ be the homogeneous component in $H^{|\alpha|+2k}(X^{[n]})$ of

$$G(\alpha, n) = p_{1*}(\text{ch}(\mathcal{O}_{Z_n}) \cdot p_2^* \alpha \cdot p_2^* \text{td}(X)) \in H^*(X^{[n]}) \quad (2.1)$$

where $\text{ch}(\mathcal{O}_{Z_n})$ denotes the Chern character of \mathcal{O}_{Z_n} and $\text{td}(X)$ denotes the Todd class. We extend the notion $G_k(\alpha, n)$ linearly to an arbitrary class $\alpha \in H^*(X)$, and set $G(\alpha, 0) = 0$. It was proved in [LQW1] that the cohomology ring of $X^{[n]}$ is generated by the classes $G_k(\alpha, n)$ where $0 \leq k < n$ and α runs over a linear basis of $H^*(X)$. The *Chern character operator* $\mathfrak{G}_k(\alpha) \in \text{End}(\mathbb{H}_X)$ is the operator acting on $H^*(X^{[n]})$ by the cup product with $G_k(\alpha, n)$. The following is from [LQW2].

Theorem 2.3. *Let $k \geq 0$ and $\alpha \in H^*(X)$. Then, $\mathfrak{G}_k(\alpha)$ is equal to*

$$\begin{aligned} & - \sum_{\ell(\lambda)=k+2, |\lambda|=0} \frac{1}{\lambda!} \mathfrak{a}_\lambda(\alpha) + \sum_{\ell(\lambda)=k, |\lambda|=0} \frac{s(\lambda) - 2}{24\lambda!} \mathfrak{a}_\lambda(e_X \alpha) \\ & + \sum_{\ell(\lambda)=k+1, |\lambda|=0} \frac{g_{1,\lambda}}{\lambda!} \mathfrak{a}_\lambda(K_X \alpha) + \sum_{\ell(\lambda)=k, |\lambda|=0} \frac{g_{2,\lambda}}{\lambda!} \mathfrak{a}_\lambda(K_X^2 \alpha) \end{aligned}$$

where all the numbers $g_{1,\lambda}$ and $g_{2,\lambda}$ are independent of X and α .

For $\alpha_1, \dots, \alpha_N \in H^*(X)$ and integers $k_1, \dots, k_N \geq 0$, define the series

$$F_{k_1, \dots, k_N}^{\alpha_1, \dots, \alpha_N}(q) = \sum_n q^n \int_{X^{[n]}} \left(\prod_{i=1}^N G_{k_i}(\alpha_i, n) \right) c(T_{X^{[n]}}). \quad (2.2)$$

In view of Göttsche's Theorem in [Got], we have $F(q) = (q; q)_\infty^{-\chi(X)}$.

The following is from [LQW3] and will be used throughout the paper.

Lemma 2.4. *Let $k, s \geq 1$, $n_1, \dots, n_k, m_1, \dots, m_s \in \mathbb{Z}$, and $\alpha, \beta \in H^*(X)$.*

(i) *The commutator $[(\mathfrak{a}_{n_1} \cdots \mathfrak{a}_{n_k})(\alpha), (\mathfrak{a}_{m_1} \cdots \mathfrak{a}_{m_s})(\beta)]$ is equal to*

$$- \sum_{t=1}^k \sum_{j=1}^s n_t \delta_{n_t, -m_j} \cdot \left(\prod_{\ell=1}^{j-1} \mathfrak{a}_{m_\ell} \prod_{1 \leq u \leq k, u \neq t} \mathfrak{a}_{n_u} \prod_{\ell=j+1}^s \mathfrak{a}_{m_\ell} \right) (\alpha \beta).$$

(ii) *Let j satisfy $1 \leq j < k$. Then, $(\mathfrak{a}_{n_1} \cdots \mathfrak{a}_{n_k})(\alpha)$ is equal to*

$$\left(\prod_{1 \leq s < j} \mathfrak{a}_{n_s} \cdot \mathfrak{a}_{n_{j+1}} \mathfrak{a}_{n_j} \cdot \prod_{j+1 < s \leq k} \mathfrak{a}_{n_s} \right) (\alpha) - n_j \delta_{n_j, -n_{j+1}} \left(\prod_{\substack{1 \leq s \leq k \\ s \neq j, j+1}} \mathfrak{a}_{n_s} \right) (e_X \alpha).$$

3. The vertex operators of Carlsson and Okounkov

In this section, we will recall the vertex operators constructed in [CO, Car], and use them to rewrite the generating series $F_{k_1, \dots, k_N}^{\alpha_1, \dots, \alpha_N}(q)$ defined in (2.2).

Let L be a line bundle over the smooth projective surface X . Let \mathbb{E}_L be the virtual vector bundle on $X^{[k]} \times X^{[\ell]}$ whose fiber at $(I, J) \in X^{[k]} \times X^{[\ell]}$ is given by

$$\mathbb{E}_L|_{(I, J)} = \chi(\mathcal{O}, L) - \chi(J, I \otimes L).$$

Let \mathfrak{L}_m be the trivial line bundle on X with a scaling action of \mathbb{C}^* of character m , and let Δ_n be the diagonal in $X^{[n]} \times X^{[n]}$. Then,

$$\mathbb{E}_{\mathfrak{L}_m}|_{\Delta_n} = T_{X^{[n]}, m}, \quad (3.1)$$

the tangent bundle $T_{X^{[n]}}$ with a scaling action of \mathbb{C}^* of character m . By abusing notations, we also use L to denote its first Chern class. Put

$$\Gamma_{\pm}(L, z) = \exp \left(\sum_{n>0} \frac{z^{\mp n}}{n} \mathfrak{a}_{\pm n}(L) \right). \quad (3.2)$$

Remark 3.1. There is a sign difference between the Heisenberg commutation relations used in [Car] (see p.3 there) and in this paper (see Theorem 2.1). So for $n > 0$, our Heisenberg operators $\mathfrak{a}_{-n}(L)$ and $\mathfrak{a}_n(-L)$ are equal to the Heisenberg operators $\mathfrak{a}_{-n}(L)$ and $\mathfrak{a}_n(L)$ in [Car]. Accordingly, our operators $\Gamma_{-}(L, z)$ and $\Gamma_{+}(-L, z)$ are equal to the operators $\Gamma_{-}(L, z)$ and $\Gamma_{+}(L, z)$ in [Car].

The following commutation relations can be found in [Car] (see Remark 3.1):

$$[\Gamma_{+}(L, x), \Gamma_{+}(L', y)] = [\Gamma_{-}(L, x), \Gamma_{-}(L', y)] = 0, \quad (3.3)$$

$$\Gamma_{+}(L, x) \Gamma_{-}(L', y) = (1 - y/x)^{\langle L, L' \rangle} \Gamma_{-}(L', y) \Gamma_{+}(L, x). \quad (3.4)$$

Let $W(L, z) : \mathbb{H}_X \rightarrow \mathbb{H}_X$ be the vertex operator constructed in [CO, Car] where z is a formal variable. By [Car], $W(L, z)$ is defined via the pairing

$$\langle W(L, z) \eta, \xi \rangle = \int_{X^{[k]} \times X^{[\ell]}} (\eta \otimes \xi) c_{k+\ell}(\mathbb{E}_L) \quad (3.5)$$

for $\eta \in H^*(X^{[k]})$ and $\xi \in H^*(X^{[\ell]})$. The main result in [Car] is (see Remark 3.1):

$$W(L, z) = \Gamma_{-}(L - K_X, z) \Gamma_{+}(-L, z). \quad (3.6)$$

Lemma 3.2. *Let \mathfrak{d} be the number-of-points operator, i.e., $\mathfrak{d}|_{H^*(X^{[n]})} = n \text{Id}$. Then,*

$$F_{k_1, \dots, k_N}^{\alpha_1, \dots, \alpha_N}(q) = \text{Tr } q^{\mathfrak{d}} W(\mathfrak{L}_1, z) \prod_{i=1}^N \mathfrak{G}_{k_i}(\alpha_i). \quad (3.7)$$

Proof. We will show that the coefficients of q^n on both sides of (3.7) are equal. Let $\{e_j\}_j$ be a linear basis of $H^*(X^{[n]})$. Then the fundamental class of the diagonal Δ_n in $X^{[n]} \times X^{[n]}$ is given by $[\Delta_n] = \sum_j (-1)^{|e_j|} e_j \otimes e_j^*$ where $\{e_j^*\}_j$ is the linear basis of $H^*(X^{[n]})$ dual to $\{e_j\}_j$ in the sense that $\langle e_j, e_{j'}^* \rangle = \delta_{j, j'}$. By the definitions

of $W(L, z)$ and $\mathfrak{G}_k(\alpha)$, $\text{Tr } q^n W(\mathfrak{L}_1, z) \prod_{i=1}^N \mathfrak{G}_{k_i}(\alpha_i)$ is equal to

$$\begin{aligned}
& q^n \sum_j (-1)^{|e_j|} \left\langle W(\mathfrak{L}_1, z) \left(\prod_{i=1}^N \mathfrak{G}_{k_i}(\alpha_i) \right) e_j, e_j^* \right\rangle \\
&= q^n \sum_j (-1)^{|e_j|} \int_{X^{[n]} \times X^{[n]}} \left(\left(\prod_{i=1}^N \mathfrak{G}_{k_i}(\alpha_i) \right) e_j \otimes e_j^* \right) c_{2n}(\mathbb{E}_{\mathfrak{L}_1}) \\
&= q^n \sum_j (-1)^{|e_j|} \int_{X^{[n]} \times X^{[n]}} \left(\left(\prod_{i=1}^N G_{k_i}(\alpha_i, n) \right) e_j \otimes e_j^* \right) c_{2n}(\mathbb{E}_{\mathfrak{L}_1}) \\
&= q^n \sum_j (-1)^{|e_j|} \int_{X^{[n]} \times X^{[n]}} (e_j \otimes e_j^*) p_1^* \left(\prod_{i=1}^N G_{k_i}(\alpha_i, n) \right) c_{2n}(\mathbb{E}_{\mathfrak{L}_1}) \\
&= q^n \int_{X^{[n]} \times X^{[n]}} [\Delta_n] p_1^* \left(\prod_{i=1}^N G_{k_i}(\alpha_i, n) \right) c_{2n}(\mathbb{E}_{\mathfrak{L}_1})
\end{aligned}$$

where $p_1 : X^{[n]} \times X^{[n]} \rightarrow X^{[n]}$ denotes the first projection. By (3.1), we have $c_{2n}(\mathbb{E}_{\mathfrak{L}_1})|_{\Delta_n} = c(T_{X^{[n]}})$. Here and below, we implicitly set $t = 1$ for the generator t of the equivariant cohomology $H_{\mathbb{C}^*}^*(\text{pt})$ of a point. Therefore,

$$\text{Tr } q^n W(\mathfrak{L}_1, z) \prod_{i=1}^N \mathfrak{G}_{k_i}(\alpha_i) = q^n \int_{X^{[n]}} \left(\prod_{i=1}^N G_{k_i}(\alpha_i, n) \right) c(T_{X^{[n]}}). \quad \square$$

4. The trace $\text{Tr } q^\partial W(\mathfrak{L}_1, z) \prod_{i=1}^N \frac{\mathfrak{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}$ and the series $F_{k_1, \dots, k_N}^{\alpha_1, \dots, \alpha_N}(q)$

In this section, we will first determine the structure of $\text{Tr } q^\partial W(\mathfrak{L}_1, z) \prod_{i=1}^N \frac{\mathfrak{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}$. Then, the structure of the generating series $F_{k_1, \dots, k_N}^{\alpha_1, \dots, \alpha_N}(q)$ will follow from Lemma 3.2, Theorem 2.3 and the structure of $\text{Tr } q^\partial W(\mathfrak{L}_1, z) \prod_{i=1}^N \frac{\mathfrak{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}$.

We begin with four technical lemmas. To explain the ideas behind these lemmas, note from (3.6) that $\text{Tr } q^\partial W(\mathfrak{L}_1, z) \prod_{i=1}^N \frac{\mathfrak{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}$ is equal to

$$\text{Tr } q^\partial \Gamma_-(\mathfrak{L}_1 - K_X, z) \Gamma_+(-\mathfrak{L}_1, z) \prod_{i=1}^N \frac{\mathfrak{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}. \quad (4.1)$$

Lemma 4.1 deals with the commutator between $\frac{\mathbf{a}_\lambda(\alpha)}{\lambda!}$ and $\exp\left(\frac{z^n}{n}\mathbf{a}_{-n}(\gamma)\right)$. It enables us in Lemma 4.2 to eliminate $\Gamma_-(\mathfrak{L}_1 - K_X, z)$ from (4.1), and allows us in Lemma 4.3 to eliminate $\Gamma_+(-\mathfrak{L}_1, z)$ from (4.1). Lemma 4.4 determines the structure of $\text{Tr } q^\diamond \prod_{i=1}^N \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}$. The proofs of these lemmas are standard but lengthy.

Recall from Definition 2.2 (ii) that $\tilde{\mathcal{P}}$ denotes the set of generalized partitions. If $\lambda = (\dots(-2)^{s-2}(-1)^{s-1}1^{s_1}2^{s_2}\dots)$ and $\mu = (\dots(-2)^{t-2}(-1)^{t-1}1^{t_1}2^{t_2}\dots)$, let

$$\lambda - \mu = (\dots(-2)^{s-2-t-2}(-1)^{s-1-t-1}1^{s_1-t_1}2^{s_2-t_2}\dots)$$

with the convention that $\lambda - \mu = \emptyset$ if $s_i < t_i$ for some i .

Lemma 4.1. *Let $\lambda \in \tilde{\mathcal{P}}$. Assume that $\gamma \in H^{\text{even}}(X)$. Then,*

$$\frac{\mathbf{a}_\lambda(\alpha)}{\lambda!} \exp\left(\frac{z^n}{n}\mathbf{a}_{-n}(\gamma)\right) = \exp\left(\frac{z^n}{n}\mathbf{a}_{-n}(\gamma)\right) \cdot \sum_{i \geq 0} \frac{(-z^n)^i}{i!} \frac{\mathbf{a}_{\lambda-(n^i)}(\gamma^i \alpha)}{(\lambda - (n^i))!}, \quad (4.2)$$

$$\exp\left(\frac{z^n}{n}\mathbf{a}_n(\gamma)\right) \cdot \frac{\mathbf{a}_\lambda(\alpha)}{\lambda!} = \sum_{i \geq 0} \frac{(-z^n)^i}{i!} \frac{\mathbf{a}_{\lambda-((-n)^i)}(\gamma^i \alpha)}{(\lambda - ((-n)^i))!} \exp\left(\frac{z^n}{n}\mathbf{a}_n(\gamma)\right). \quad (4.3)$$

Proof. Note that the adjoint of $\mathbf{a}_m(\beta)$ is equal to $(-1)^m \mathbf{a}_{-m}(\beta)$. So (4.3) follows from (4.2) by taking adjoint on both sides of (4.2) and by making suitable adjustments. To prove (4.2), put $A = \frac{\mathbf{a}_\lambda(\alpha)}{\lambda!} \exp\left(\frac{z^n}{n}\mathbf{a}_{-n}(\gamma)\right)$. Then,

$$\begin{aligned} A &= \frac{\mathbf{a}_\lambda(\alpha)}{\lambda!} \sum_{t \geq 0} \frac{1}{t!} \left(\frac{z^n}{n}\mathbf{a}_{-n}(\gamma)\right)^t \\ &= \frac{1}{\lambda!} \sum_{t \geq 0} \frac{1}{t!} \left(\frac{z^n}{n}\right)^t \sum_{i=0}^t \binom{t}{i} (\mathbf{a}_{-n}(\gamma))^{t-i} [\dots [\mathbf{a}_\lambda(\alpha), \underbrace{\mathbf{a}_{-n}(\gamma), \dots, \mathbf{a}_{-n}(\gamma)}_{i \text{ times}}]]. \end{aligned}$$

Let $\lambda = (\dots(-2)^{s-2}(-1)^{s-1}1^{s_1}2^{s_2}\dots)$. We conclude from Lemma 2.4 (i) that the commutator $[\dots [\mathbf{a}_\lambda(\alpha), \underbrace{\mathbf{a}_{-n}(\gamma), \dots, \mathbf{a}_{-n}(\gamma)}_{i \text{ times}}]]$ is equal to

$$s_n(s_n - 1) \cdots (s_n + 1 - i) (-n)^i \mathbf{a}_{\lambda-(n^i)}(\gamma^i \alpha)$$

where by our convention, $\lambda - (n^i) = \emptyset$ if $s_n < i$. So A is equal to

$$\begin{aligned} &\frac{1}{\lambda!} \sum_{t \geq 0} \frac{1}{t!} \left(\frac{z^n}{n}\right)^t \sum_{i=0}^t \binom{t}{i} (\mathbf{a}_{-n}(\gamma))^{t-i} \cdot \\ &\quad \cdot s_n(s_n - 1) \cdots (s_n + 1 - i) (-n)^i \mathbf{a}_{\lambda-(n^i)}(\gamma^i \alpha). \end{aligned}$$

Simplifying this, we complete the proof of our formula (4.2). \square

Let $\tilde{\mathcal{P}}_+ = \mathcal{P}$ be the subset of $\tilde{\mathcal{P}}$ consisting of the usual partitions, and $\tilde{\mathcal{P}}_-$ be the subset of $\tilde{\mathcal{P}}$ consisting of generalized partitions of the form $(\dots(-2)^{s-2}(-1)^{s-1})$.

Lemma 4.2. Let $\lambda^{(1)}, \dots, \lambda^{(N)} \in \tilde{\mathcal{P}}$ be generalized partitions, and $\alpha_1, \dots, \alpha_N \in H^*(X)$. Then, the trace $\text{Tr } q^\partial W(\mathfrak{L}_1, z) \prod_{i=1}^N \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}$ is equal to

$$\sum_{\substack{\mu^{(i,s)} \in \tilde{\mathcal{P}}_+ \\ 1 \leq i \leq N, s \geq 1}} \prod_{\substack{1 \leq i \leq N \\ s, n \geq 1}} \frac{(-(zq^s)^n)^{m_n^{(i,s)}}}{m_n^{(i,s)}!} \cdot \text{Tr } q^\partial \Gamma_+(-\mathfrak{L}_1, z) \prod_{i=1}^N \frac{\mathbf{a}_{\lambda^{(i)} - \sum_{s \geq 1} \mu^{(i,s)}}((1_X - K_X)^{\sum_{s, n \geq 1} m_n^{(i,s)}} \alpha_i)}{(\lambda^{(i)} - \sum_{s \geq 1} \mu^{(i,s)})!}. \quad (4.4)$$

where $\mu^{(i,s)} = (1^{m_1^{(i,s)}} \dots n^{m_n^{(i,s)}} \dots) \in \tilde{\mathcal{P}}_+$ for $1 \leq i \leq N$ and $s \geq 1$.

Proof. For simplicity, put $Q_1 = \text{Tr } q^\partial W(\mathfrak{L}_1, z) \prod_{i=1}^N \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}$. By (3.6),

$$\begin{aligned} Q_1 &= \text{Tr } q^\partial \Gamma_-(\mathfrak{L}_1 - K_X, z) \Gamma_+(-\mathfrak{L}_1, z) \prod_{i=1}^N \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \\ &= \text{Tr } \Gamma_-(\mathfrak{L}_1 - K_X, zq) q^\partial \Gamma_+(-\mathfrak{L}_1, z) \prod_{i=1}^N \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \\ &= \text{Tr } q^\partial \Gamma_+(-\mathfrak{L}_1, z) \prod_{i=1}^N \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \cdot \Gamma_-(\mathfrak{L}_1 - K_X, zq). \end{aligned} \quad (4.5)$$

By (3.2) and applying (4.2) repeatedly, we obtain

$$\begin{aligned} &\prod_{i=1}^N \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \cdot \Gamma_-(\mathfrak{L}_1 - K_X, zq) \\ &= \prod_{i=1}^N \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \cdot \exp \left(\sum_{n>0} \frac{(zq)^n}{n} \mathbf{a}_{-n}(\mathfrak{L}_1 - K_X) \right) \\ &= \Gamma_-(\mathfrak{L}_1 - K_X, zq) \sum_{\substack{\mu^{(i,1)} \in \tilde{\mathcal{P}}_+ \\ 1 \leq i \leq N}} \prod_{\substack{1 \leq i \leq N \\ n \geq 1}} \frac{(-(zq)^n)^{m_n^{(i,1)}}}{m_n^{(i,1)}!} \\ &\quad \cdot \prod_{i=1}^N \frac{\mathbf{a}_{\lambda^{(i)} - \mu^{(i,1)}}((1_X - K_X)^{\sum_{n \geq 1} m_n^{(i,1)}} \alpha_i)}{(\lambda^{(i)} - \mu^{(i,1)})!}. \end{aligned}$$

where $\mu^{(i,1)} = (1^{m_1^{(i,1)}} \dots n^{m_n^{(i,1)}} \dots)$. Therefore, Q_1 is equal to

$$\text{Tr } q^\partial \Gamma_+(-\mathfrak{L}_1, z) \Gamma_-(\mathfrak{L}_1 - K_X, zq) \sum_{\substack{\mu^{(i,1)} \in \tilde{\mathcal{P}}_+ \\ 1 \leq i \leq N}} \prod_{\substack{1 \leq i \leq N \\ n \geq 1}} \frac{(-(zq)^n)^{m_n^{(i,1)}}}{m_n^{(i,1)}!}.$$

$$\cdot \prod_{i=1}^N \frac{\mathbf{a}_{\lambda^{(i)} - \mu^{(i,1)}}((1_X - K_X)^{\sum_{n \geq 1} m_n^{(i,1)}} \alpha_i)}{(\lambda^{(i)} - \mu^{(i,1)})!}.$$

Since $\langle \mathfrak{L}_1, \mathfrak{L}_1 - K_X \rangle = 0$, we see from (3.4) that Q_1 is equal to

$$\begin{aligned} & \text{Tr } q^\partial \Gamma_- (\mathfrak{L}_1 - K_X, zq) \Gamma_+ (-\mathfrak{L}_1, z) \cdot \sum_{\substack{\mu^{(i,1)} \in \tilde{\mathcal{P}}_+ \\ 1 \leq i \leq N}} \prod_{\substack{1 \leq i \leq N \\ n \geq 1}} \frac{(-(zq)^n)^{m_n^{(i,1)}}}{m_n^{(i,1)}!} \\ & \cdot \prod_{i=1}^N \frac{\mathbf{a}_{\lambda^{(i)} - \mu^{(i,1)}}((1_X - K_X)^{\sum_{n \geq 1} m_n^{(i,1)}} \alpha_i)}{(\lambda^{(i)} - \mu^{(i,1)})!}. \end{aligned}$$

Repeat the above process beginning at line (4.5) s times. Then, Q_1 is equals to

$$\begin{aligned} & \text{Tr } q^\partial \Gamma_- (\mathfrak{L}_1 - K_X, zq^s) \Gamma_+ (-\mathfrak{L}_1, z) \cdot \sum_{\substack{\mu^{(i,r)} \in \tilde{\mathcal{P}}_+ \\ 1 \leq i \leq N, 1 \leq r \leq s}} \prod_{\substack{1 \leq i \leq N \\ 1 \leq r \leq s, n \geq 1}} \frac{(-(zq^r)^n)^{m_n^{(i,r)}}}{m_n^{(i,r)}!} \\ & \cdot \prod_{i=1}^N \frac{\mathbf{a}_{\lambda^{(i)} - \sum_{r=1}^s \mu^{(i,r)}}((1_X - K_X)^{\sum_{r=1}^s \sum_{n \geq 1} m_n^{(i,r)}} \alpha_i)}{(\lambda^{(i)} - \sum_{r=1}^s \mu^{(i,r)})!} \end{aligned}$$

where $\mu^{(i,r)} = (1^{m_1^{(i,r)}} \dots n^{m_n^{(i,r)}} \dots)$. Letting $s \rightarrow +\infty$ proves our lemma. \square

Lemma 4.3. *Let $\tilde{\lambda}^{(1)}, \dots, \tilde{\lambda}^{(N)} \in \tilde{\mathcal{P}}$ be generalized partitions, and $\tilde{\alpha}_1, \dots, \tilde{\alpha}_N \in H^*(X)$. Then, the trace $\text{Tr } q^\partial \Gamma_+ (-\mathfrak{L}_1, z) \prod_{i=1}^N \frac{\mathbf{a}_{\tilde{\lambda}^{(i)}}(\tilde{\alpha}_i)}{\tilde{\lambda}^{(i)}!}$ is equal to*

$$\sum_{\sum_{i=1}^N (|\tilde{\lambda}^{(i)}| - \sum_{t \geq 1} |\tilde{\mu}^{(i,t)}|) = 0} \prod_{\substack{1 \leq i \leq N \\ t, n \geq 1}} \frac{(z^{-1} q^{t-1})^{n \tilde{m}_n^{(i,t)}}}{\tilde{m}_n^{(i,t)}!} \cdot \text{Tr } q^\partial \prod_{i=1}^N \frac{\mathbf{a}_{\tilde{\lambda}^{(i)} - \sum_{t \geq 1} \tilde{\mu}^{(i,t)}}(\tilde{\alpha}_i)}{(\tilde{\lambda}^{(i)} - \sum_{t \geq 1} \tilde{\mu}^{(i,t)})!}$$

where $\tilde{\mu}^{(i,t)} = (\dots (-n)^{\tilde{m}_n^{(i,t)}} \dots (-1)^{\tilde{m}_1^{(i,t)}}) \in \tilde{\mathcal{P}}_-$ for $1 \leq i \leq N$ and $t \geq 1$.

Proof. For simplicity, put $Q_2 = \text{Tr } q^\partial \Gamma_+ (-\mathfrak{L}_1, z) \prod_{i=1}^N \frac{\mathbf{a}_{\tilde{\lambda}^{(i)}}(\tilde{\alpha}_i)}{\tilde{\lambda}^{(i)}!}$. By (3.2),

$$Q_2 = \text{Tr } q^\partial \exp \left(\sum_{n > 0} \frac{z^{-n}}{n} \mathbf{a}_n (-\mathfrak{L}_1) \right) \prod_{i=1}^N \frac{\mathbf{a}_{\tilde{\lambda}^{(i)}}(\tilde{\alpha}_i)}{\tilde{\lambda}^{(i)}!}.$$

Applying (4.3) repeatedly, we see that Q_2 is equal to

$$\sum_{\substack{\tilde{\mu}^{(i,1)} \in \tilde{\mathcal{P}}_- \\ 1 \leq i \leq N}} \prod_{\substack{1 \leq i \leq N \\ n \geq 1}} \frac{z^{-n \tilde{m}_n^{(i,1)}}}{\tilde{m}_n^{(i,1)}!} \cdot \text{Tr } q^\partial \prod_{i=1}^N \frac{\mathbf{a}_{\tilde{\lambda}^{(i)} - \tilde{\mu}^{(i,1)}}(\tilde{\alpha}_i)}{(\tilde{\lambda}^{(i)} - \tilde{\mu}^{(i,1)})!} \cdot \Gamma_+ (-\mathfrak{L}_1, z)$$

where $\tilde{\mu}^{(i,1)} = (\dots(-n)^{\tilde{m}_n^{(i,1)}} \dots (-1)^{\tilde{m}_1^{(i,1)}}) \in \tilde{\mathcal{P}}_-$. Now Q_2 is equal to

$$\begin{aligned} & \sum_{\substack{\tilde{\mu}^{(i,1)} \in \tilde{\mathcal{P}}_- \\ 1 \leq i \leq N}} \prod_{\substack{1 \leq i \leq N \\ n \geq 1}} \frac{z^{-n\tilde{m}_n^{(i,1)}}}{\tilde{m}_n^{(i,1)}!} \cdot q^{\sum_{i=1}^N (|\tilde{\mu}^{(i,1)}| - |\tilde{\lambda}^{(i)}|)} \text{Tr} \prod_{i=1}^N \frac{\mathbf{a}_{\tilde{\lambda}^{(i)} - \tilde{\mu}^{(i,1)}}(\tilde{\alpha}_i)}{(\tilde{\lambda}^{(i)} - \tilde{\mu}^{(i,1)})!} \cdot q^{\mathfrak{d}\Gamma_+(-\mathfrak{L}_1, z)} \\ &= \sum_{\substack{\tilde{\mu}^{(i,1)} \in \tilde{\mathcal{P}}_- \\ 1 \leq i \leq N}} \prod_{\substack{1 \leq i \leq N \\ n \geq 1}} \frac{z^{-n\tilde{m}_n^{(i,1)}}}{\tilde{m}_n^{(i,1)}!} \cdot q^{\sum_{i=1}^N (|\tilde{\mu}^{(i,1)}| - |\tilde{\lambda}^{(i)}|)} \text{Tr} q^{\mathfrak{d}\Gamma_+(-\mathfrak{L}_1, z)} \prod_{i=1}^N \frac{\mathbf{a}_{\tilde{\lambda}^{(i)} - \tilde{\mu}^{(i,1)}}(\tilde{\alpha}_i)}{(\tilde{\lambda}^{(i)} - \tilde{\mu}^{(i,1)})!}. \end{aligned}$$

By degree reason, $\text{Tr} q^{\mathfrak{d}\Gamma_+(-\mathfrak{L}_1, z)} \mathbf{a}_{\mu}(\beta) = 0$ if $|\mu| > 0$. If $|\mu| = 0$, then we have $\text{Tr} q^{\mathfrak{d}\Gamma_+(-\mathfrak{L}_1, z)} \mathbf{a}_{\mu}(\beta) = \text{Tr} q^{\mathfrak{d}} \mathbf{a}_{\mu}(\beta)$. So Q_2 is equal to

$$\begin{aligned} & \sum_{\substack{\sum_{i=1}^N (|\tilde{\lambda}^{(i)}| - |\tilde{\mu}^{(i,1)}|) < 0 \\ 1 \leq i \leq N \\ n \geq 1}} \prod_{\substack{1 \leq i \leq N \\ n \geq 1}} \frac{z^{-n\tilde{m}_n^{(i,1)}}}{\tilde{m}_n^{(i,1)}!} \cdot q^{\sum_{i=1}^N (|\tilde{\mu}^{(i,1)}| - |\tilde{\lambda}^{(i)}|)} \text{Tr} q^{\mathfrak{d}\Gamma_+(-\mathfrak{L}_1, z)} \prod_{i=1}^N \frac{\mathbf{a}_{\tilde{\lambda}^{(i)} - \tilde{\mu}^{(i,1)}}(\tilde{\alpha}_i)}{(\tilde{\lambda}^{(i)} - \tilde{\mu}^{(i,1)})!} \\ &+ \sum_{\substack{\sum_{i=1}^N (|\tilde{\lambda}^{(i)}| - |\tilde{\mu}^{(i,1)}|) = 0 \\ 1 \leq i \leq N \\ n \geq 1}} \prod_{\substack{1 \leq i \leq N \\ n \geq 1}} \frac{z^{-n\tilde{m}_n^{(i,1)}}}{\tilde{m}_n^{(i,1)}!} \cdot \text{Tr} q^{\mathfrak{d}} \prod_{i=1}^N \frac{\mathbf{a}_{\tilde{\lambda}^{(i)} - \tilde{\mu}^{(i,1)}}(\tilde{\alpha}_i)}{(\tilde{\lambda}^{(i)} - \tilde{\mu}^{(i,1)})!}. \end{aligned}$$

Repeating the process in the previous paragraph t times, we conclude that

$$Q_2 = U(t) - V(t)$$

where $U(t)$ is given by

$$\sum_{\substack{\sum_{i=1}^N (|\tilde{\lambda}^{(i)}| - \sum_{r=1}^t |\tilde{\mu}^{(i,r)}|) < 0 \\ 1 \leq i \leq N \\ n \geq 1}} \prod_{r=1}^t \left(\prod_{\substack{1 \leq i \leq N \\ n \geq 1}} \frac{z^{-n\tilde{m}_n^{(i,r)}}}{\tilde{m}_n^{(i,r)}!} \cdot q^{\sum_{i=1}^N (\sum_{\ell=1}^r |\tilde{\mu}^{(i,\ell)}| - |\tilde{\lambda}^{(i)}|)} \right). \quad (4.6)$$

$$\cdot \text{Tr} q^{\mathfrak{d}\Gamma_+(-\mathfrak{L}_1, z)} \prod_{i=1}^N \frac{\mathbf{a}_{\tilde{\lambda}^{(i)} - \sum_{r=1}^t \tilde{\mu}^{(i,r)}}(\tilde{\alpha}_i)}{(\tilde{\lambda}^{(i)} - \sum_{r=1}^t \tilde{\mu}^{(i,r)})!} \quad (4.7)$$

with $\tilde{\mu}^{(i,r)} = (\dots(-n)^{\tilde{m}_n^{(i,r)}} \dots (-1)^{\tilde{m}_1^{(i,r)}}) \in \tilde{\mathcal{P}}_-$, and $V(t)$ is given by

$$\begin{aligned} & \sum_{\substack{\sum_{i=1}^N (|\tilde{\lambda}^{(i)}| - \sum_{r=1}^t |\tilde{\mu}^{(i,r)}|) = 0 \\ 1 \leq i \leq N \\ n \geq 1}} \prod_{r=1}^t \left(\prod_{\substack{1 \leq i \leq N \\ n \geq 1}} \frac{z^{-n\tilde{m}_n^{(i,r)}}}{\tilde{m}_n^{(i,r)}!} \cdot q^{\sum_{i=1}^N (\sum_{\ell=1}^r |\tilde{\mu}^{(i,\ell)}| - |\tilde{\lambda}^{(i)}|)} \right) \\ & \cdot \text{Tr} q^{\mathfrak{d}} \prod_{i=1}^N \frac{\mathbf{a}_{\tilde{\lambda}^{(i)} - \sum_{r=1}^t \tilde{\mu}^{(i,r)}}(\tilde{\alpha}_i)}{(\tilde{\lambda}^{(i)} - \sum_{r=1}^t \tilde{\mu}^{(i,r)})!}. \end{aligned}$$

Denote line (4.6) by $\tilde{U}(t)$. Since $\sum_{i=1}^N (|\tilde{\lambda}^{(i)}| - \sum_{r=1}^t |\tilde{\mu}^{(i,r)}|) < 0$ and $|\tilde{\mu}^{(i,r)}| < 0$, $\tilde{U}(t)$ is a polynomial in q with coefficients being bounded in terms of $-\sum_{i=1}^N |\tilde{\lambda}^{(i)}|$.

Moreover, $q^t |\tilde{U}(t)|$. Line (4.7) is bounded in terms of the generalized partitions $\tilde{\lambda}^{(i)}$. Since $0 < q < 1$, $U(t) \rightarrow 0$ as $t \rightarrow +\infty$. Letting $t \rightarrow +\infty$, we see that Q_2 equals

$$\sum_{\sum_{i=1}^N (|\tilde{\lambda}^{(i)}| - \sum_{t \geq 1} |\tilde{\mu}^{(i,t)}|) = 0} \prod_{t \geq 1} \left(\prod_{\substack{1 \leq i \leq N \\ n \geq 1}} \frac{z^{-n \tilde{m}_n^{(i,t)}}}{\tilde{m}_n^{(i,t)}!} \cdot q^{\sum_{i=1}^N (\sum_{\ell=1}^t |\tilde{\mu}^{(i,\ell)}| - |\tilde{\lambda}^{(i)}|)} \right).$$

$$\cdot \text{Tr } q^{\mathfrak{d}} \prod_{i=1}^N \frac{\mathfrak{a}_{\tilde{\lambda}^{(i)} - \sum_{t \geq 1} \tilde{\mu}^{(i,t)}}(\tilde{\alpha}_i)}{(\tilde{\lambda}^{(i)} - \sum_{t \geq 1} \tilde{\mu}^{(i,t)})!}.$$

Replacing $q^{\sum_{i=1}^N (\sum_{\ell=1}^t |\tilde{\mu}^{(i,\ell)}| - |\tilde{\lambda}^{(i)}|)}$ by $q^{-\sum_{i=1}^N \sum_{\ell \geq t+1} |\tilde{\mu}^{(i,\ell)}|}$ proves our lemma. \square

Lemma 4.4. *Let $\lambda^{(1)}, \dots, \lambda^{(N)} \in \tilde{\mathcal{P}}$ be generalized partitions, and $\alpha_1, \dots, \alpha_N \in H^*(X)$ be homogeneous. Then, $\text{Tr } q^{\mathfrak{d}} \prod_{i=1}^N \frac{\mathfrak{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}$ can be computed by an induction on N , and is a linear combination of expressions of the form:*

$$(q; q)_{\infty}^{-\chi(X)} \cdot \text{Sign}(\pi) \cdot \prod_{i=1}^u \left\langle e_X^{m_i}, \prod_{j \in \pi_i} \alpha_j \right\rangle \cdot \prod_{i=1}^v \frac{n_i^{k_i} q^{n_i}}{1 - q^{n_i}} \quad (4.8)$$

where $0 \leq v \leq \sum_{i=1}^N \ell(\lambda^{(i)})/2$, $m_i \geq 0$, $n_i > 0$, all the integers involved and the partition $\{\pi_1, \dots, \pi_u\}$ of $\{1, \dots, N\}$ depend only on $\lambda^{(1)}, \dots, \lambda^{(N)}$, and $\text{Sign}(\pi)$ is the sign compensating the formal difference between $\prod_{i=1}^u \prod_{j \in \pi_i} \alpha_j$ and $\alpha_1 \cdots \alpha_N$. Moreover, the coefficients of this linear combination are independent of q, α_i, n_i, X .

Proof. For simplicity, put $A_N = \text{Tr } q^{\mathfrak{d}} \prod_{i=1}^N \frac{\mathfrak{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}$. Since $\mathfrak{a}_{\lambda^{(i)}}(\alpha_i)$ has conformal weight $|\lambda^{(i)}|$, $A_N = 0$ unless $\sum_{i=1}^N |\lambda^{(i)}| = 0$. In the rest of the proof, we will assume $\sum_{i=1}^N |\lambda^{(i)}| = 0$. We will divide the proof into two cases.

Case 1: $|\lambda^{(i)}| = 0$ for every $1 \leq i \leq N$. Then, $\ell(\lambda^{(i)}) \geq 2$ for every i . Since $\mathfrak{a}_{\lambda^{(i)}}(\alpha_i)$ has degree $2(\ell(\lambda^{(i)}) - 2) + |\alpha_i|$, $A_N = 0$ unless $\ell(\lambda^{(i)}) = 2$ and $|\alpha_i| = 0$ for all $1 \leq i \leq N$. Assume that $\ell(\lambda^{(i)}) = 2$ and $|\alpha_i| = 0$ for all $1 \leq i \leq N$. Then for every $1 \leq i \leq N$, we have $\lambda^{(i)} = ((-n_i)n_i)$ for some $n_i > 0$. We further assume that $n_1 = \dots = n_r$ for some $1 \leq r \leq N$ and $n_i \neq n_1$ if $r < i \leq N$. Let $\alpha_1 = a1_X$

and $\tau_{2*}1_X = \sum_j (-1)^{|\beta_j|} \beta_j \otimes \gamma_j$ with $\langle \beta_j, \gamma_{j'} \rangle = \delta_{j,j'}$. Then, A_N is equal to

$$\begin{aligned}
& a \sum_j (-1)^{|\beta_j|} \text{Tr } q^\partial \mathbf{a}_{-n_1}(\beta_j) \mathbf{a}_{n_1}(\gamma_j) \prod_{i=2}^N \mathbf{a}_{\lambda(i)}(\alpha_i) \\
&= a q^{n_1} \sum_j (-1)^{|\beta_j|} \text{Tr } \mathbf{a}_{-n_1}(\beta_j) q^\partial \mathbf{a}_{n_1}(\gamma_j) \prod_{i=2}^N \mathbf{a}_{\lambda(i)}(\alpha_i) \\
&= a q^{n_1} \sum_j \text{Tr } q^\partial \mathbf{a}_{n_1}(\gamma_j) \prod_{i=2}^N \mathbf{a}_{\lambda(i)}(\alpha_i) \cdot \mathbf{a}_{-n_1}(\beta_j) \\
&= a q^{n_1} \sum_j \text{Tr } q^\partial \mathbf{a}_{n_1}(\gamma_j) \mathbf{a}_{-n_1}(\beta_j) \prod_{i=2}^N \mathbf{a}_{\lambda(i)}(\alpha_i) \\
&\quad + a q^{n_1} \sum_j \sum_{i=2}^r \text{Tr } q^\partial \mathbf{a}_{n_1}(\gamma_j) \prod_{k=2}^{i-1} \mathbf{a}_{\lambda(k)}(\alpha_k) \cdot [\mathbf{a}_{\lambda(i)}(\alpha_i), \mathbf{a}_{-n_1}(\beta_j)] \cdot \prod_{k=i+1}^N \mathbf{a}_{\lambda(k)}(\alpha_k).
\end{aligned}$$

By Lemma 2.4 (i), A_N is equal to the sum of the expressions

$$\left\langle e_X, \alpha_1 \prod_{i=1}^{k_1} \alpha_{j_i} \right\rangle \cdot \frac{(-n_1)^{k_1} q^{n_1}}{1 - q^{n_1}} \cdot \text{Tr } q^\partial \prod_{i \in \{2, \dots, N\} - \{j_1, \dots, j_{k_1}\}} \mathbf{a}_{\lambda(i)}(\alpha_i) \quad (4.9)$$

where $0 \leq k_1 \leq r-1$, $\{j_1, \dots, j_{k_1}\} \subset \{2, \dots, r\}$, every factor in $(-n_1)^{k_1}$ comes from a commutator of type $[\mathbf{a}_{n_1}(\cdot), \mathbf{a}_{-n_1}(\cdot)]$, and the coefficients of this linear combination depend only on $\lambda^{(1)}, \dots, \lambda^{(N)}$. In particular, we have

$$A_1 = \text{Tr } q^\partial \mathbf{a}_{\lambda(1)}(\alpha_1) = (q; q)_\infty^{-\chi(X)} \cdot \langle e_X, \alpha_1 \rangle \cdot \frac{(-n_1) q^{n_1}}{1 - q^{n_1}}. \quad (4.10)$$

Combining with (4.9), we see that our lemma holds in this case.

Case 2: $\sum_{i=1}^N |\lambda^{(i)}| = 0$ but $|\lambda^{(i_0)}| \neq 0$ for some i_0 . Then, $N \geq 2$, and we may assume that $|\lambda^{(i_0)}| < 0$. To simplify the expressions, we further assume that every α_i has an even degree. Note that A_N can be rewritten as

$$\begin{aligned}
& \text{Tr } q^\partial \frac{\mathbf{a}_{\lambda(i_0)}(\alpha_{i_0})}{\lambda^{(i_0)}!} \prod_{1 \leq i \leq N, i \neq i_0} \frac{\mathbf{a}_{\lambda(i)}(\alpha_i)}{\lambda^{(i)}!} \\
&+ \sum_{r=1}^{i_0-1} \text{Tr } q^\partial \prod_{i=1}^{r-1} \frac{\mathbf{a}_{\lambda(i)}(\alpha_i)}{\lambda^{(i)}!} \cdot \left[\frac{\mathbf{a}_{\lambda(r)}(\alpha_r)}{\lambda^{(r)}!}, \frac{\mathbf{a}_{\lambda(i_0)}(\alpha_{i_0})}{\lambda^{(i_0)}!} \right] \cdot \prod_{r+1 \leq i \leq N, i \neq i_0} \frac{\mathbf{a}_{\lambda(i)}(\alpha_i)}{\lambda^{(i)}!}.
\end{aligned}$$

Since $q^\flat \mathbf{a}_{\lambda^{(i_0)}}(\alpha_{i_0}) = q^{-|\lambda^{(i_0)}|} \mathbf{a}_{\lambda^{(i_0)}}(\alpha_{i_0}) q^\flat$, we see that A_N is equal to

$$\begin{aligned} & q^{-|\lambda^{(i_0)}|} \operatorname{Tr} q^\flat \frac{\mathbf{a}_{\lambda^{(i_0)}}(\alpha_{i_0})}{\lambda^{(i_0)}!} q^\flat \prod_{1 \leq i \leq N, i \neq i_0} \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \\ & + \sum_{r=1}^{i_0-1} \operatorname{Tr} q^\flat \prod_{i=1}^{r-1} \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \cdot \left[\frac{\mathbf{a}_{\lambda^{(r)}}(\alpha_r)}{\lambda^{(r)}!}, \frac{\mathbf{a}_{\lambda^{(i_0)}}(\alpha_{i_0})}{\lambda^{(i_0)}!} \right] \cdot \prod_{r+1 \leq i \leq N, i \neq i_0} \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \\ & = q^{-|\lambda^{(i_0)}|} \operatorname{Tr} q^\flat \prod_{1 \leq i \leq N, i \neq i_0} \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \cdot \frac{\mathbf{a}_{\lambda^{(i_0)}}(\alpha_{i_0})}{\lambda^{(i_0)}!} \\ & + \sum_{r=1}^{i_0-1} \operatorname{Tr} q^\flat \prod_{i=1}^{r-1} \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \cdot \left[\frac{\mathbf{a}_{\lambda^{(r)}}(\alpha_r)}{\lambda^{(r)}!}, \frac{\mathbf{a}_{\lambda^{(i_0)}}(\alpha_{i_0})}{\lambda^{(i_0)}!} \right] \cdot \prod_{r+1 \leq i \leq N, i \neq i_0} \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}. \end{aligned}$$

Note that $\operatorname{Tr} q^\flat \prod_{1 \leq i \leq N, i \neq i_0} \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \cdot \frac{\mathbf{a}_{\lambda^{(i_0)}}(\alpha_{i_0})}{\lambda^{(i_0)}!}$ is equal to

$$A_N + \sum_{r=i_0+1}^N \operatorname{Tr} q^\flat \prod_{1 \leq i \leq r-1, i \neq i_0} \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \cdot \left[\frac{\mathbf{a}_{\lambda^{(r)}}(\alpha_r)}{\lambda^{(r)}!}, \frac{\mathbf{a}_{\lambda^{(i_0)}}(\alpha_{i_0})}{\lambda^{(i_0)}!} \right] \cdot \prod_{i=r+1}^N \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}.$$

Therefore, we conclude that $(1 - q^{-|\lambda^{(i_0)}|})A_N$ is equal to

$$\begin{aligned} & q^{-|\lambda^{(i_0)}|} \sum_{r=i_0+1}^N \operatorname{Tr} q^\flat \prod_{1 \leq i \leq r-1, i \neq i_0} \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \cdot \left[\frac{\mathbf{a}_{\lambda^{(r)}}(\alpha_r)}{\lambda^{(r)}!}, \frac{\mathbf{a}_{\lambda^{(i_0)}}(\alpha_{i_0})}{\lambda^{(i_0)}!} \right] \cdot \prod_{i=r+1}^N \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \\ & + \sum_{r=1}^{i_0-1} \operatorname{Tr} q^\flat \prod_{i=1}^{r-1} \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \cdot \left[\frac{\mathbf{a}_{\lambda^{(r)}}(\alpha_r)}{\lambda^{(r)}!}, \frac{\mathbf{a}_{\lambda^{(i_0)}}(\alpha_{i_0})}{\lambda^{(i_0)}!} \right] \cdot \prod_{r+1 \leq i \leq N, i \neq i_0} \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}. \end{aligned}$$

Put $n_0 = -|\lambda^{(i_0)}| > 0$. It follows that A_N is equal to

$$\begin{aligned} & \frac{q^{n_0}}{1 - q^{n_0}} \sum_{r=i_0+1}^N \operatorname{Tr} q^\flat \prod_{1 \leq i \leq r-1, i \neq i_0} \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \cdot \left[\frac{\mathbf{a}_{\lambda^{(r)}}(\alpha_r)}{\lambda^{(r)}!}, \frac{\mathbf{a}_{\lambda^{(i_0)}}(\alpha_{i_0})}{\lambda^{(i_0)}!} \right] \cdot \prod_{i=r+1}^N \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \\ & + \frac{1}{1 - q^{n_0}} \sum_{r=1}^{i_0-1} \operatorname{Tr} q^\flat \prod_{i=1}^{r-1} \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \cdot \left[\frac{\mathbf{a}_{\lambda^{(r)}}(\alpha_r)}{\lambda^{(r)}!}, \frac{\mathbf{a}_{\lambda^{(i_0)}}(\alpha_{i_0})}{\lambda^{(i_0)}!} \right] \cdot \prod_{r+1 \leq i \leq N, i \neq i_0} \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}. \quad (4.11) \end{aligned}$$

By Lemma 2.4 (i) and (ii), our lemma holds in this case as well. \square

The following theorem provides the structure of the trace

$$\operatorname{Tr} q^\flat W(\mathfrak{L}_1, z) \prod_{i=1}^N \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}.$$

Theorem 4.5. For $1 \leq i \leq N$, let $\lambda^{(i)} = (\dots (-n)^{\tilde{m}_n^{(i)}} \dots (-1)^{\tilde{m}_1^{(i)}} 1^{m_1^{(i)}} \dots n^{m_n^{(i)}} \dots)$ and $\alpha_i \in H^*(X)$ be homogeneous. Then, $\text{Tr } q^\partial W(\mathfrak{L}_1, z) \prod_{i=1}^N \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}$ is equal to

$$z^{\sum_{i=1}^N |\lambda^{(i)}|} \cdot (q; q)_\infty^{-\chi(X)} \cdot \prod_{i=1}^N \langle (1_X - K_X)^{\sum_{n \geq 1} m_n^{(i)}}, \alpha_i \rangle \cdot \prod_{1 \leq i \leq N, n \geq 1} \left(\frac{(-1)^{m_n^{(i)}}}{m_n^{(i)}!} \frac{q^{nm_n^{(i)}}}{(1 - q^n)^{m_n^{(i)}}} \frac{1}{\tilde{m}_n^{(i)}!} \frac{1}{(1 - q^n)^{\tilde{m}_n^{(i)}}} \right) + \widetilde{W},$$

and the lower weight term \widetilde{W} is a linear combination of expressions of the form:

$$z^{\sum_{i=1}^N |\lambda^{(i)}|} \cdot (q; q)_\infty^{-\chi(X)} \cdot \text{Sign}(\pi) \cdot \prod_{i=1}^u \left\langle K_X^{r_i} e_X^{r'_i}, \prod_{j \in \pi_i} \alpha_j \right\rangle \cdot \prod_{i=1}^v \frac{q^{n_i w_i p_i}}{(1 - q^{n_i})^{w_i}} \quad (4.12)$$

where $\sum_{i=1}^v w_i < \sum_{i=1}^N \ell(\lambda^{(i)})$, the integers $u, v, r_i, r'_i \geq 0, n_i > 0, w_i > 0, p_i \in \{0, 1\}$ and the partition $\pi = \{\pi_1, \dots, \pi_u\}$ of $\{1, \dots, N\}$ depend only on the generalized partitions $\lambda^{(1)}, \dots, \lambda^{(N)}$, and $\text{Sign}(\pi)$ is the sign compensating the formal difference between $\prod_{i=1}^u \prod_{j \in \pi_i} \alpha_j$ and $\alpha_1 \cdots \alpha_N$. Moreover, the coefficients of this linear combination are independent of $q, \alpha_1, \dots, \alpha_N$ and X .

Proof. For simplicity, put $\text{Tr}_\lambda = \text{Tr } q^\partial W(\mathfrak{L}_1, z) \prod_{i=1}^N \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}$. Combining Lemma 4.2 and Lemma 4.3, we conclude that Tr_λ is equal to

$$\sum_{\substack{\sum_{i=1}^N (|\lambda^{(i)}| - \sum_{s \geq 1} |\mu^{(i,s)}| - \sum_{t \geq 1} |\tilde{\mu}^{(i,t)}|) = 0 \\ \mu^{(i,s)} \in \tilde{\mathcal{P}}_+, \tilde{\mu}^{(i,t)} \in \tilde{\mathcal{P}}_-}} \prod_{\substack{1 \leq i \leq N \\ s, n \geq 1}} \frac{(-zq^s)^{m_n^{(i,s)}}}{m_n^{(i,s)}!} \cdot \prod_{\substack{1 \leq i \leq N \\ t, n \geq 1}} \frac{(z^{-1}q^{t-1})^{n\tilde{m}_n^{(i,t)}}}{\tilde{m}_n^{(i,t)}!} \cdot \text{Tr } q^\partial \prod_{i=1}^N \frac{\mathbf{a}_{\lambda^{(i)} - \sum_{s \geq 1} \mu^{(i,s)} - \sum_{t \geq 1} \tilde{\mu}^{(i,t)}}((1_X - K_X)^{\sum_{s, n \geq 1} m_n^{(i,s)}} \alpha_i)}{(\lambda^{(i)} - \sum_{s \geq 1} \mu^{(i,s)} - \sum_{t \geq 1} \tilde{\mu}^{(i,t)})!}$$

where $\mu^{(i,s)} = (1^{m_1^{(i,s)}} \dots n^{m_n^{(i,s)}} \dots)$ and $\tilde{\mu}^{(i,t)} = (\dots (-n)^{\tilde{m}_n^{(i,t)}} \dots (-1)^{\tilde{m}_1^{(i,t)}})$. The sum of all the exponents of z is $\sum_{i=1}^N |\lambda^{(i)}|$. So Tr_λ is equal to

$$z^{\sum_{i=1}^N |\lambda^{(i)}|} \cdot \sum_{\substack{\sum_{i=1}^N (|\lambda^{(i)}| - \sum_{s \geq 1} |\mu^{(i,s)}| - \sum_{t \geq 1} |\tilde{\mu}^{(i,t)}|) = 0 \\ \mu^{(i,s)} \in \tilde{\mathcal{P}}_+, \tilde{\mu}^{(i,t)} \in \tilde{\mathcal{P}}_-}} \prod_{\substack{1 \leq i \leq N \\ s, n \geq 1}} \frac{(-q^{sn})^{m_n^{(i,s)}}}{m_n^{(i,s)}!} \cdot \prod_{\substack{1 \leq i \leq N \\ t, n \geq 1}} \frac{q^{(t-1)n\tilde{m}_n^{(i,t)}}}{\tilde{m}_n^{(i,t)}!} \cdot \text{Tr } q^\partial \prod_{i=1}^N \frac{\mathbf{a}_{\lambda^{(i)} - \sum_{s \geq 1} \mu^{(i,s)} - \sum_{t \geq 1} \tilde{\mu}^{(i,t)}}((1_X - K_X)^{\sum_{s, n \geq 1} m_n^{(i,s)}} \alpha_i)}{(\lambda^{(i)} - \sum_{s \geq 1} \mu^{(i,s)} - \sum_{t \geq 1} \tilde{\mu}^{(i,t)})!} \quad (4.13)$$

By our convention, $\sum_{s \geq 1} \mu^{(i,s)} + \sum_{t \geq 1} \tilde{\mu}^{(i,t)} \leq \lambda^{(i)}$ for every $1 \leq i \leq N$. We now divide the rest of the proof into Case A and Case B.

Case A: $\sum_{s \geq 1} \mu^{(i,s)} + \sum_{t \geq 1} \tilde{\mu}^{(i,t)} = \lambda^{(i)}$ for every $1 \leq i \leq N$. Then line (4.13) is

$$\mathrm{Tr} q^{\mathfrak{d}} \cdot \prod_{i=1}^N \langle (1_X - K_X)^{\sum_{s,n \geq 1} m_n^{(i,s)}}, \alpha_i \rangle = (q; q)_{\infty}^{-\chi(X)} \prod_{i=1}^N \langle (1_X - K_X)^{\sum_{s,n \geq 1} m_n^{(i,s)}}, \alpha_i \rangle.$$

Therefore, the contribution C_1 of this case to Tr_{λ} is equal to

$$z^{\sum_{i=1}^N |\lambda^{(i)}|} \cdot (q; q)_{\infty}^{-\chi(X)} \prod_{i=1}^N \langle (1_X - K_X)^{\sum_{n \geq 1} m_n^{(i)}}, \alpha_i \rangle \cdot \sum_{\substack{\sum_{s \geq 1} m_n^{(i,s)} = m_n^{(i)} \\ 1 \leq i \leq N, n \geq 1}} \prod_{\substack{1 \leq i \leq N \\ s, n \geq 1}} \frac{(-q^{sn})^{m_n^{(i,s)}}}{m_n^{(i,s)}!} \cdot \sum_{\substack{\sum_{t \geq 1} \tilde{m}_n^{(i,t)} = \tilde{m}_n^{(i)} \\ 1 \leq i \leq N, n \geq 1}} \prod_{\substack{1 \leq i \leq N \\ t, n \geq 1}} \frac{q^{(t-1)n\tilde{m}_n^{(i,t)}}}{\tilde{m}_n^{(i,t)}!}.$$

Rewrite q^{sn} as $q^{(s-1)n}q^n$. Then the contribution C_1 is equal to

$$z^{\sum_{i=1}^N |\lambda^{(i)}|} \cdot (q; q)_{\infty}^{-\chi(X)} \prod_{i=1}^N \langle (1_X - K_X)^{\sum_{n \geq 1} m_n^{(i)}}, \alpha_i \rangle \cdot \prod_{1 \leq i \leq N, n \geq 1} (-q^n)^{m_n^{(i)}} \cdot \sum_{\substack{\sum_{s \geq 1} m_n^{(i,s)} = m_n^{(i)} \\ 1 \leq i \leq N, n \geq 1}} \prod_{\substack{1 \leq i \leq N \\ s, n \geq 1}} \frac{q^{(s-1)n m_n^{(i,s)}}}{m_n^{(i,s)}!} \cdot \sum_{\substack{\sum_{t \geq 1} \tilde{m}_n^{(i,t)} = \tilde{m}_n^{(i)} \\ 1 \leq i \leq N, n \geq 1}} \prod_{\substack{1 \leq i \leq N \\ t, n \geq 1}} \frac{q^{(t-1)n \tilde{m}_n^{(i,t)}}}{\tilde{m}_n^{(i,t)}!}.$$

Since $\sum_{\substack{\sum_{s \geq 1} m_n^{(i,s)} = m_n^{(i)} \\ 1 \leq i \leq N, n \geq 1}} \prod_{s, n \geq 1} \frac{(q^{(s-1)n})^{i_{s,n}}}{i_{s,n}!} = \prod_{n \geq 1} \left(\frac{1}{i_n!} \frac{1}{(1 - q^n)^{i_n}} \right)$, C_1 is equal to

$$z^{\sum_{i=1}^N |\lambda^{(i)}|} \cdot (q; q)_{\infty}^{-\chi(X)} \prod_{i=1}^N \langle (1_X - K_X)^{\sum_{n \geq 1} m_n^{(i)}}, \alpha_i \rangle \cdot \prod_{1 \leq i \leq N, n \geq 1} \left(\frac{(-1)^{m_n^{(i)}}}{m_n^{(i)}!} \frac{q^{n m_n^{(i)}}}{(1 - q^n)^{m_n^{(i)}}} \right) \cdot \prod_{1 \leq i \leq N, n \geq 1} \left(\frac{1}{\tilde{m}_n^{(i)}!} \frac{1}{(1 - q^n)^{\tilde{m}_n^{(i)}}} \right). \quad (4.14)$$

Case B: $\sum_{s \geq 1} \mu^{(i,s)} + \sum_{t \geq 1} \tilde{\mu}^{(i,t)} < \lambda^{(i)}$ for some $1 \leq i \leq N$. Without loss of generality, we may assume that $\sum_{s \geq 1} \mu^{(i,s)} + \sum_{t \geq 1} \tilde{\mu}^{(i,t)} = \lambda^{(i)}$ for every $1 \leq i \leq N_1$ where $N_1 < N$, and $\sum_{s \geq 1} \mu^{(i,s)} + \sum_{t \geq 1} \tilde{\mu}^{(i,t)} < \lambda^{(i)}$ for every $N_1 + 1 \leq i \leq N$. For $N_1 + 1 \leq i \leq N$, put $\sum_{s \geq 1} \mu^{(i,s)} + \sum_{t \geq 1} \tilde{\mu}^{(i,t)} = \tilde{\lambda}^{(i)} = (\dots (-n)^{\tilde{p}_n^{(i)}} \dots (-1)^{\tilde{p}_1^{(i)}} 1^{p_1^{(i)}} \dots n^{p_n^{(i)}} \dots)$. An argument similar to that in the previous paragraph shows that for the fixed generalized partitions $\tilde{\lambda}^{(i)}$ with $N_1 + 1 \leq i \leq N$, the contribution C_2 of this case to Tr_{λ} is equal to

$$z^{\sum_{i=1}^N |\lambda^{(i)}|} \cdot \prod_{i=1}^{N_1} \langle (1_X - K_X)^{\sum_{n \geq 1} m_n^{(i)}}, \alpha_i \rangle \cdot \prod_{\substack{1 \leq i \leq N_1 \\ n \geq 1}} \left(\frac{(-1)^{m_n^{(i)}}}{m_n^{(i)}!} \frac{q^{n m_n^{(i)}}}{(1 - q^n)^{m_n^{(i)}}} \frac{1}{\tilde{m}_n^{(i)}!} \frac{1}{(1 - q^n)^{\tilde{m}_n^{(i)}}} \right)$$

$$\begin{aligned}
& \cdot \prod_{\substack{N_1+1 \leq i \leq N \\ n \geq 1}} \left(\frac{(-1)^{p_n^{(i)}}}{p_n^{(i)}!} \frac{q^{np_n^{(i)}}}{(1-q^n)^{p_n^{(i)}}} \frac{1}{\tilde{p}_n^{(i)}!} \frac{1}{(1-q^n)^{\tilde{p}_n^{(i)}}} \right) \\
& \cdot \text{Tr } q^\partial \prod_{i=N_1+1}^N \frac{\mathbf{a}_{\lambda^{(i)}-\tilde{\lambda}^{(i)}}((1_X - K_X)^{\sum_{n \geq 1} p_n^{(i)}} \alpha_i)}{(\lambda^{(i)} - \tilde{\lambda}^{(i)})!}. \tag{4.15}
\end{aligned}$$

By Lemma 4.4, C_2 is a linear combination of expressions of the form:

$$\begin{aligned}
& z^{\sum_{i=1}^N |\lambda^{(i)}|} \cdot \prod_{i=1}^{N_1} \langle (1_X - K_X)^{\sum_{n \geq 1} m_n^{(i)}} \alpha_i \rangle \cdot \prod_{\substack{1 \leq i \leq N_1 \\ n \geq 1}} \left(\frac{(-1)^{m_n^{(i)}}}{m_n^{(i)}!} \frac{q^{nm_n^{(i)}}}{(1-q^n)^{m_n^{(i)}}} \frac{1}{\tilde{m}_n^{(i)}!} \frac{1}{(1-q^n)^{\tilde{m}_n^{(i)}}} \right) \\
& \cdot \prod_{\substack{N_1+1 \leq i \leq N \\ n \geq 1}} \left(\frac{(-1)^{p_n^{(i)}}}{p_n^{(i)}!} \frac{q^{np_n^{(i)}}}{(1-q^n)^{p_n^{(i)}}} \frac{1}{\tilde{p}_n^{(i)}!} \frac{1}{(1-q^n)^{\tilde{p}_n^{(i)}}} \right) \cdot \\
& (q; q)_\infty^{-\chi(X)} \cdot \text{Sign}(\pi) \cdot \prod_{i=1}^u \left\langle e_X^{m_i}, \prod_{j \in \pi_i} ((1_X - K_X)^{\sum_{n \geq 1} p_n^{(j)}} \alpha_j) \right\rangle \cdot \prod_{i=1}^v \frac{q^{n_i}}{1 - q^{n_i}}
\end{aligned}$$

where $v < \sum_{i=N_1+1}^N \ell(\lambda^{(i)} - \tilde{\lambda}^{(i)})$, $n_i > 0$, $m_i \geq 0$, $\{\pi_1, \dots, \pi_u\}$ is a partition of $\{N_1+1, \dots, N\}$, and $\text{Sign}(\pi)$ compensates the formal difference between $\prod_{i=1}^u \prod_{j \in \pi_i} \alpha_j$ and $\alpha_{N_1+1} \cdots \alpha_N$. The coefficients of this linear combination are independent of $q, \alpha_1, \dots, \alpha_N$ and X , and depend only on the partitions $\lambda^{(i)} - \tilde{\lambda}^{(i)}$. Note that for nonnegative integers a and b , the pairing $\langle e_X^a, (1_X - K_X)^b \beta \rangle = \langle e_X^a (1_X - K_X)^b, \beta \rangle$ is a linear combination of $\langle e_X^a K_X^c, \beta \rangle$, $0 \leq c \leq b$. In addition, we have

$$\sum_{1 \leq i \leq N_1, n \geq 1} (m_n^{(i)} + \tilde{m}_n^{(i)}) + \sum_{N_1+1 \leq i \leq N, n \geq 1} (p_n^{(i)} + \tilde{p}_n^{(i)}) + v < \sum_{i=1}^N \ell(\lambda^{(i)})$$

regarding the weights in C_2 . It follows that C_2 is a linear combination of the expressions (4.12). Combining with (4.14) completes the proof of our theorem. \square

Remark 4.6. When $N = 1$, we can work out the lower weight term \widetilde{W} in Theorem 4.5 by examining its proof more carefully and by using (4.10). To state the result, let $\lambda = (\cdots (-n)^{\tilde{m}_n} \cdots (-1)^{\tilde{m}_1} 1^{m_1} \cdots n^{m_n} \cdots) \in \widetilde{\mathcal{P}}$. For $n_1 \geq 1$ with $m_{n_1} \cdot \tilde{m}_{n_1} \geq 1$, define $m_{n_1}(n_1) = m_{n_1} - 1$, $\tilde{m}_{n_1}(n_1) = \tilde{m}_{n_1} - 1$, and $m_n(n_1) = m_n$

and $\tilde{m}_n(n_1) = \tilde{m}_n$ if $n \neq n_1$. Then, $\text{Tr } q^\partial W(\mathfrak{L}_1, z) \frac{\mathbf{a}_\lambda(\alpha)}{\lambda!}$ is equal to the sum

$$\begin{aligned} & z^{|\lambda|} \cdot (q; q)_\infty^{-\chi(X)} \cdot \langle (1_X - K_X)^{\sum_{n \geq 1} m_n}, \alpha \rangle \cdot \\ & \cdot \prod_{n \geq 1} \left(\frac{(-1)^{m_n}}{m_n!} \frac{q^{nm_n}}{(1 - q^n)^{m_n}} \frac{1}{\tilde{m}_n!} \frac{1}{(1 - q^n)^{\tilde{m}_n}} \right) \\ & + z^{|\lambda|} \cdot (q; q)_\infty^{-\chi(X)} \cdot \langle e_X, \alpha \rangle \cdot \sum_{n_1 \geq 1 \text{ with } m_{n_1} \cdot \tilde{m}_{n_1} \geq 1} \frac{n_1 q^{n_1}}{1 - q^{n_1}} \cdot \\ & \cdot \prod_{n \geq 1} \left(\frac{(-1)^{m_n(n_1)}}{m_n(n_1)!} \frac{q^{nm_n(n_1)}}{(1 - q^n)^{m_n(n_1)}} \frac{1}{\tilde{m}_n(n_1)!} \frac{1}{(1 - q^n)^{\tilde{m}_n(n_1)}} \right). \end{aligned}$$

The next lemma is used to organize the leading term in Theorem 4.5.

Lemma 4.7. *For $\alpha \in H^*(X)$ and $k \geq 0$, define $\Theta_k^\alpha(q)$ to be*

$$- \sum_{\ell(\lambda)=k+2, |\lambda|=0} \langle (1_X - K_X)^{\sum_{n \geq 1} i_n}, \alpha \rangle \cdot \prod_{n \geq 1} \left(\frac{(-1)^{i_n}}{i_n!} \frac{q^{ni_n}}{(1 - q^n)^{i_n}} \frac{1}{\tilde{i}_n!} \frac{1}{(1 - q^n)^{\tilde{i}_n}} \right) \quad (4.16)$$

where $\lambda = (\dots (-n)^{\tilde{i}_n} \dots (-1)^{\tilde{i}_1} 1^{i_1} \dots n^{i_n} \dots)$. Then, $\Theta_k^\alpha(q) = \text{Coeff}_{z^0} \Theta_k^\alpha(q, z)$ which denotes the coefficient of z^0 in $\Theta_k^\alpha(q, z)$ defined by

$$\begin{aligned} & - \sum_{\substack{a, s_1, \dots, s_a, b, t_1, \dots, t_b \geq 1 \\ \sum_{i=1}^a s_i + \sum_{j=1}^b t_j = k+2}} \langle (1_X - K_X)^{\sum_{i=1}^a s_i}, \alpha \rangle \prod_{i=1}^a \frac{(-1)^{s_i}}{s_i!} \cdot \prod_{j=1}^b \frac{1}{t_j!} \\ & \cdot \sum_{n_1 > \dots > n_a} \prod_{i=1}^a \frac{(qz)^{n_i s_i}}{(1 - q^{n_i})^{s_i}} \cdot \sum_{m_1 > \dots > m_b} \prod_{j=1}^b \frac{z^{-m_j t_j}}{(1 - q^{m_j})^{t_j}}. \end{aligned} \quad (4.17)$$

Proof. Put $A = \langle (1_X - K_X)^{\sum_{n \geq 1} i_n}, \alpha \rangle$ which implicitly depends on $\sum_{n \geq 1} i_n$. Rewrite $|\lambda|$ and $\ell(\lambda)$ in terms of the integers i_n and \tilde{i}_n . Then, $\Theta_k^\alpha(q)$ is equal to

$$- \sum_{\substack{\sum_{n \geq 1} i_n + \sum_{n \geq 1} \tilde{i}_n = k+2 \\ \sum_{n \geq 1} n i_n = \sum_{n \geq 1} n \tilde{i}_n > 0}} A \prod_{n \geq 1} \left(\frac{(-1)^{i_n}}{i_n!} \frac{q^{ni_n}}{(1 - q^n)^{i_n}} \frac{1}{\tilde{i}_n!} \frac{1}{(1 - q^n)^{\tilde{i}_n}} \right). \quad (4.18)$$

Denote the *positive* integers in the *ordered* list $\{i_1, \dots, i_n, \dots\}$ by s_a, \dots, s_1 respectively (e.g., if the ordered list $\{i_1, \dots, i_n, \dots\}$ is $\{2, 0, 5, 4, 0, \dots\}$, then $a = 3$ with $s_3 = 2, s_2 = 5, s_1 = 4$). We have $a \geq 1$. Similarly, denote the *positive* integers in the *ordered* list $\{\tilde{i}_1, \dots, \tilde{i}_n, \dots\}$ by t_b, \dots, t_1 respectively. Then $b \geq 1$. Since $\sum_{n \geq 1} i_n = \sum_{i=1}^a s_i$, we get $A = \langle (1_X - K_X)^{\sum_{i=1}^a s_i}, \alpha \rangle$. Rewriting (4.18) in terms of s_a, \dots, s_1 and t_b, \dots, t_1 , we see that $\Theta_k^\alpha(q) = \text{Coeff}_{z^0} \Theta_k^\alpha(q, z)$. \square

We remark that the multiple q -zeta value $\Theta_k^\alpha(q, z)$ has weight $(k + 2)$.

Theorem 4.8. *For $1 \leq i \leq N$, let $k_i \geq 0$ and $\alpha_i \in H^*(X)$ be homogeneous. Then,*

$$F_{k_1, \dots, k_N}^{\alpha_1, \dots, \alpha_N}(q) = (q; q)_\infty^{-\chi(X)} \cdot \text{Coeff}_{z_1^0 \dots z_N^0} \left(\prod_{i=1}^N \Theta_{k_i}^{\alpha_i}(q, z_i) \right) + W_1, \quad (4.19)$$

and the lower weight term W_1 is an infinite linear combination of the expressions:

$$(q; q)_\infty^{-\chi(X)} \cdot \text{Sign}(\pi) \cdot \prod_{i=1}^u \left\langle K_X^{r_i} e_X^{r'_i}, \prod_{j \in \pi_i} \alpha_j \right\rangle \cdot \prod_{i=1}^v \frac{q^{n_i w_i p_i}}{(1 - q^{n_i})^{w_i}} \quad (4.20)$$

where $\sum_{i=1}^v w_i < \sum_{i=1}^N (k_i + 2)$, and the integers $u, v, r_i, r'_i \geq 0, n_i > 0, w_i > 0, p_i \in \{0, 1\}$ and the partition $\pi = \{\pi_1, \dots, \pi_u\}$ of $\{1, \dots, N\}$ depend only on the integers k_i . Moreover, the coefficients of this linear combination are independent of q, α_i, X .

Proof. By Lemma 3.2, $F_{k_1, \dots, k_N}^{\alpha_1, \dots, \alpha_N}(q) = \text{Tr } q^\partial W(\mathfrak{L}_1, z) \prod_{i=1}^N \mathfrak{G}_{k_i}(\alpha_i)$. Combining with Theorem 2.3 and Theorem 4.5, we conclude that

$$F_{k_1, \dots, k_N}^{\alpha_1, \dots, \alpha_N}(q) = \tilde{F}_{k_1, \dots, k_N}^{\alpha_1, \dots, \alpha_N}(q) + W_{1,1} \quad (4.21)$$

where $W_{1,1}$ is an infinite linear combination of the expressions (4.20), and

$$\tilde{F}_{k_1, \dots, k_N}^{\alpha_1, \dots, \alpha_N}(q) := (-1)^N \cdot \sum_{\substack{\ell(\lambda^{(i)}) = k_i + 2, |\lambda^{(i)}| = 0 \\ 1 \leq i \leq N}} \text{Tr } q^\partial W(\mathfrak{L}_1, z) \prod_{i=1}^N \frac{\mathfrak{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}. \quad (4.22)$$

Applying Theorem 4.5 again, we see that $\tilde{F}_{k_1, \dots, k_N}^{\alpha_1, \dots, \alpha_N}(q)$ is equal to

$$\begin{aligned} & (-1)^N (q; q)_\infty^{-\chi(X)} \cdot \sum_{\substack{\ell(\lambda^{(i)}) = k_i + 2, |\lambda^{(i)}| = 0 \\ 1 \leq i \leq N}} \prod_{i=1}^N \langle (1_X - K_X)^{\sum_{n \geq 1} m_n^{(i)}}, \alpha_i \rangle \\ & \cdot \prod_{1 \leq i \leq N, n \geq 1} \left(\frac{(-1)^{m_n^{(i)}}}{m_n^{(i)}!} \frac{q^{n m_n^{(i)}}}{(1 - q^n)^{m_n^{(i)}}} \frac{1}{\tilde{m}_n^{(i)}!} \frac{1}{(1 - q^n)^{\tilde{m}_n^{(i)}}} \right) + W_{1,2} \end{aligned} \quad (4.23)$$

where the lower weight term $W_{1,2}$ is an infinite linear combination of the expressions (4.20), and we have put $\lambda^{(i)} = (\dots (-n)^{\tilde{m}_n^{(i)}} \dots (-1)^{\tilde{m}_1^{(i)}} 1^{m_1^{(i)}} \dots n^{m_n^{(i)}} \dots)$. So

$$\begin{aligned} \tilde{F}_{k_1, \dots, k_N}^{\alpha_1, \dots, \alpha_N}(q) &= (q; q)_\infty^{-\chi(X)} \cdot \prod_{i=1}^N \Theta_{k_i}^{\alpha_i}(q) + W_{1,2} \\ &= (q; q)_\infty^{-\chi(X)} \cdot \text{Coeff}_{z_1^0 \dots z_N^0} \left(\prod_{i=1}^N \Theta_{k_i}^{\alpha_i}(q, z_i) \right) + W_{1,2} \end{aligned} \quad (4.24)$$

by Lemma 4.7. Putting $W_1 = W_{1,1} + W_{1,2}$ completes the proof of (4.19). \square

Our next goal is to relate the lower weight term $W_{1,2}$ in (4.23) and (4.24) to multiple q -zeta values (with additional variables z_1, \dots, z_N inserted). We will assume $e_X \alpha_i = 0$ for all $1 \leq i \leq N$. We begin with a lemma strengthening Lemma 4.4.

Lemma 4.9. *Let $\lambda^{(1)}, \dots, \lambda^{(N)} \in \tilde{\mathcal{P}}$, and $\alpha_1, \dots, \alpha_N \in H^*(X)$ be homogeneous. Assume that $e_X \alpha_i = 0$ for every $1 \leq i \leq N$, and $\sum_{i=1}^N |\lambda^{(i)}| = 0$. Put*

$$A_N = \text{Tr } q^\flat \prod_{i=1}^N \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}.$$

- (i) *If $\ell(\lambda^{(i)}) \geq 2$ for every $1 \leq i \leq N$, then $A_N = 0$.*
- (ii) *If $A_N \neq 0$, then A_N is a linear combination of the expressions:*

$$(q; q)_\infty^{-\chi(X)} \cdot \text{Sign}(\pi) \cdot \prod_{i=1}^u \left\langle 1_X, \prod_{j \in \pi_i} \alpha_j \right\rangle \cdot \prod_{i=1}^{\tilde{\ell}} \frac{(-\tilde{n}_i) q^{\tilde{n}_i \tilde{p}_i}}{1 - q^{\tilde{n}_i}} \quad (4.25)$$

$$= (q; q)_\infty^{-\chi(X)} \cdot \text{Sign}(\pi) \cdot \prod_{i=1}^u \left\langle 1_X, \prod_{j \in \pi_i} \alpha_j \right\rangle \cdot \prod_{i=1}^{\ell} \frac{(-n'_i)^{w_i} q^{n'_i p_i}}{(1 - q^{n'_i})^{w_i}} \quad (4.26)$$

where $\tilde{\ell} = \sum_{i=1}^N \ell(\lambda^{(i)})/2 = \sum_{i=1}^{\ell} w_i$, $\tilde{p}_i \in \{0, 1\}$, $0 \leq p_i \leq w_i$, the partition $\pi = \{\pi_1, \dots, \pi_u\}$ of $\{1, \dots, N\}$ depend only on $\lambda^{(1)}, \dots, \lambda^{(N)}$, the integers $\tilde{n}_1, \dots, \tilde{n}_{\tilde{\ell}}$ are the positive parts (repeated with multiplicities) in $\lambda^{(1)}, \dots, \lambda^{(N)}$, the integers n'_1, \dots, n'_ℓ denote the different integers in $\tilde{n}_1, \dots, \tilde{n}_{\tilde{\ell}}$, and each n'_i appears w_i times in $\tilde{n}_1, \dots, \tilde{n}_{\tilde{\ell}}$.

Proof. (i) As in the proof of Lemma 4.4, $A_N = 0$ unless $\ell(\lambda^{(i)}) = 2$ and $|\alpha_i| = 0$ for every $1 \leq i \leq N$. Assume $\ell(\lambda^{(i)}) = 2$ and $|\alpha_i| = 0$ for every $1 \leq i \leq N$. To prove $A_N = 0$, we will use induction on N . If $N = 1$, then $A_1 = 0$ by (4.10). Let $N \geq 2$. If $|\lambda^{(i)}| = 0$ for every $1 \leq i \leq N$, then $A_N = 0$ by (4.9). Assume $|\lambda^{(i_0)}| \neq 0$ for some $1 \leq i_0 \leq N$. Since $\sum_{i=1}^N |\lambda^{(i)}| = 0$, we may further assume that $|\lambda^{(i_0)}| < 0$. By (4.11), Lemma 2.4 (i) and (ii), and induction, we conclude that $A_N = 0$.

(ii) Note that (4.26) follows from (4.25) since each integer n'_i appears w_i times among the integers $\tilde{n}_1, \dots, \tilde{n}_{\tilde{\ell}}$. In the following, we will prove (4.25). To simplify the signs, we will assume that $|\alpha_i|$ is even for every i .

Since $A_N \neq 0$, we conclude from (i) that $\ell(\lambda^{(i_0)}) = 1$ for some $1 \leq i_0 \leq N$. If $\lambda^{(i_0)} = (-n_0)$ for some $n_0 > 0$, then by (4.11), A_N is equal to

$$\begin{aligned} & \frac{1}{1 - q^{n_0}} \sum_{r=1}^{i_0-1} \text{Tr } q^\flat \prod_{i=1}^{r-1} \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \cdot \left[\frac{\mathbf{a}_{\lambda^{(r)}}(\alpha_r)}{\lambda^{(r)}!}, \mathbf{a}_{-n_0}(\alpha_{i_0}) \right] \cdot \prod_{r+1 \leq i \leq N, i \neq i_0} \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \\ & + \frac{q^{n_0}}{1 - q^{n_0}} \sum_{r=i_0+1}^N \text{Tr } q^\flat \prod_{1 \leq i \leq r-1, i \neq i_0} \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \cdot \left[\frac{\mathbf{a}_{\lambda^{(r)}}(\alpha_r)}{\lambda^{(r)}!}, \mathbf{a}_{-n_0}(\alpha_{i_0}) \right] \cdot \prod_{i=r+1}^N \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}. \end{aligned}$$

Similarly, if $\lambda^{(i_0)} = (n_0)$ for some $n_0 > 0$, then A_N is equal to

$$\frac{q^{n_0}}{1 - q^{n_0}} \sum_{r=1}^{i_0-1} \text{Tr } q^\flat \prod_{i=1}^{r-1} \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \cdot \left[\mathbf{a}_{n_0}(\alpha_{i_0}), \frac{\mathbf{a}_{\lambda^{(r)}}(\alpha_r)}{\lambda^{(r)}!} \right] \cdot \prod_{r+1 \leq i \leq N, i \neq i_0} \frac{\mathbf{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}$$

$$+ \frac{1}{1 - q^{n_0}} \sum_{r=i_0+1}^N \text{Tr } q^{\mathfrak{d}} \prod_{1 \leq i \leq r-1, i \neq i_0} \frac{\mathfrak{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!} \cdot \left[\mathfrak{a}_{n_0}(\alpha_{i_0}), \frac{\mathfrak{a}_{\lambda^{(r)}}(\alpha_r)}{\lambda^{(r)}!} \right] \cdot \prod_{i=r+1}^N \frac{\mathfrak{a}_{\lambda^{(i)}}(\alpha_i)}{\lambda^{(i)}!}.$$

Note that $[\mathfrak{a}_{\lambda^{(r)}}(\alpha_r)/\lambda^{(r)}!, \mathfrak{a}_{-n_0}(\alpha_{i_0})] = (-n_0)\mathfrak{a}_{\lambda^{(r)}-(n_0)}(\alpha_r\alpha_{i_0})/(\lambda^{(r)} - (n_0))!$, and $[\mathfrak{a}_{n_0}(\alpha_{i_0}), \mathfrak{a}_{\lambda^{(r)}}(\alpha_r)/\lambda^{(r)}!] = (-n_0)\mathfrak{a}_{\lambda^{(r)}-(-n_0)}(\alpha_{i_0}\alpha_r)/(\lambda^{(r)} - (-n_0))!$. Therefore, by induction, A_N is a linear combination of the expressions (4.25). We remark that the negative parts (repeated with multiplicities) in $\lambda^{(1)}, \dots, \lambda^{(N)}$ are $-\tilde{n}_1, \dots, -\tilde{n}_{\tilde{\ell}}$. \square

Theorem 4.10. *For $1 \leq i \leq N$, let $k_i \geq 0$ and $\alpha_i \in H^*(X)$ be homogeneous. Assume that $e_X \alpha_i = 0$ for every $1 \leq i \leq N$. Then,*

$$\tilde{F}_{k_1, \dots, k_N}^{\alpha_1, \dots, \alpha_N}(q) = (q; q)_{\infty}^{-\chi(X)} \cdot \text{Coeff}_{z_1^0 \dots z_N^0} \left(\prod_{i=1}^N \Theta_{k_i}^{\alpha_i}(q, z_i) \right) + W_{1,2}, \quad (4.27)$$

and $(q; q)_{\infty}^{\chi(X)} \cdot W_{1,2}$ is a linear combination of the coefficients of $z_1^0 \dots z_N^0$ in some multiple q -zeta values (with variables z_1, \dots, z_N inserted) of weights $< \sum_{i=1}^N (k_i + 2)$. Moreover, the coefficients in this linear combination are independent of q .

Proof. To simplify the signs, we will assume that $|\alpha_i|$ is even for every i . Recall that $\tilde{F}_{k_1, \dots, k_N}^{\alpha_1, \dots, \alpha_N}(q)$ is defined in (4.22), and that (4.27) is just (4.24). From the proofs of (4.24) and Theorem 4.5, we see that the lower weight term $W_{1,2}$ in (4.27) is the contributions of Case B in the proof of Theorem 4.5 to the right-hand-side of (4.22). By (4.15) and Lemma 4.7, up to a re-ordering of the set $\{1, \dots, N\}$, these contributions are of the following form, denoted by $C_{2, N-N_1}$:

$$\begin{aligned} & \text{Coeff}_{z_1^0 \dots z_{N_1}^0} \left(\prod_{i=1}^{N_1} \Theta_{k_i}^{\alpha_i}(q, z_i) \right) \\ & \cdot (-1)^{N-N_1} \cdot \sum_{\substack{\ell(\lambda^{(i)})=k_i+2, |\lambda^{(i)}|=0 \\ N_1+1 \leq i \leq N}} \sum_{\substack{\tilde{\lambda}^{(i)} < \lambda^{(i)} \\ N_1+1 \leq i \leq N}} \prod_{\substack{N_1+1 \leq i \leq N \\ n \geq 1}} \left(\frac{(-1)^{p_n^{(i)}}}{p_n^{(i)}!} \frac{q^{np_n^{(i)}}}{(1-q^n)^{p_n^{(i)}}} \frac{1}{\tilde{p}_n^{(i)}!} \frac{1}{(1-q^n)^{\tilde{p}_n^{(i)}}} \right) \\ & \cdot \text{Tr } q^{\mathfrak{d}} \prod_{i=N_1+1}^N \frac{\mathfrak{a}_{\lambda^{(i)}-\tilde{\lambda}^{(i)}}((1_X - K_X)^{\sum_{n \geq 1} p_n^{(i)}} \alpha_i)}{(\lambda^{(i)} - \tilde{\lambda}^{(i)})!} \end{aligned}$$

where $0 \leq N_1 < N$, $\tilde{\lambda}^{(i)}$ is denoted by $(\dots (-n)^{\tilde{p}_n^{(i)}} \dots (-1)^{\tilde{p}_1^{(i)}} 1^{p_1^{(i)}} \dots n^{p_n^{(i)}} \dots)$, and $\sum_{i=N_1+1}^N |\lambda^{(i)} - \tilde{\lambda}^{(i)}| = 0$. We may let $N_1 = 0$. Put $\mu^{(i)} = \lambda^{(i)} - \tilde{\lambda}^{(i)}$. Then $C_{2, N}$ is

$$(-1)^N \cdot \sum_{\substack{\ell(\tilde{\lambda}^{(i)})+\ell(\mu^{(i)})=k_i+2 \\ |\tilde{\lambda}^{(i)}|+|\mu^{(i)}|=0 \\ 1 \leq i \leq N}} \prod_{\substack{1 \leq i \leq N \\ n \geq 1}} \left(\frac{(-1)^{p_n^{(i)}}}{p_n^{(i)}!} \frac{q^{np_n^{(i)}}}{(1-q^n)^{p_n^{(i)}}} \frac{1}{\tilde{p}_n^{(i)}!} \frac{1}{(1-q^n)^{\tilde{p}_n^{(i)}}} \right) \quad (4.28)$$

$$\cdot \text{Tr } q^{\mathfrak{d}} \prod_{i=1}^N \frac{\mathfrak{a}_{\mu^{(i)}}((1_X - K_X)^{\sum_{n \geq 1} p_n^{(i)}} \alpha_i)}{\mu^{(i)}!} \quad (4.29)$$

where $\mu^{(i)} \neq \emptyset$ for every $1 \leq i \leq N$, and $\sum_{i=1}^N |\mu^{(i)}| = 0$. By Lemma 4.9 (ii), the trace on line (4.29) is a linear combination of the expressions:

$$(q; q)_\infty^{-\chi(X)} \cdot \prod_{i=1}^u \left\langle 1_X, \prod_{j \in \pi_i} ((1_X - K_X)^{\sum_{n \geq 1} p_n^{(j)}} \alpha_j) \right\rangle \cdot \prod_{i=1}^\ell \frac{(-n'_i)^{w_i} q^{n'_i p_i}}{(1 - q^{n'_i})^{w_i}} \quad (4.30)$$

where $\sum_{i=1}^\ell w_i = \sum_{i=1}^N \ell(\mu^{(i)})/2$, $0 \leq p_i \leq w_i$, and the mutually distinct integers n'_1, \dots, n'_ℓ appear w_1, \dots, w_ℓ times respectively as the positive parts (repeated with multiplicities) of $\mu^{(1)}, \dots, \mu^{(N)}$ (so the negative parts, repeated with multiplicities, of $\mu^{(1)}, \dots, \mu^{(N)}$ are $-n'_1, \dots, -n'_\ell$ with multiplicities w_1, \dots, w_ℓ respectively).

We now fix the type of the N -tuple $(\mu^{(1)}, \dots, \mu^{(N)})$. Define \mathfrak{T} to be the set consisting of all the N -tuples $(\tilde{\mu}^{(1)}, \dots, \tilde{\mu}^{(N)})$ obtained from $(\mu^{(1)}, \dots, \mu^{(N)})$ as follows: take N mutually distinct positive integers n_1, \dots, n_ℓ , and obtain $\tilde{\mu}^{(i)}$, $1 \leq i \leq N$ from $\mu^{(i)}$, $1 \leq i \leq N$ by replacing every part $\pm n'_j$ in $\mu^{(i)}$ by $\pm n_j$. Denote the contribution of the type \mathfrak{T} to $C_{2,N}$ by $C_{2,N}^\mathfrak{T}$. Then, $C_{2,N} = \sum_{\mathfrak{T}} C_{2,N}^\mathfrak{T}$. Thus, to prove the statement about $(q; q)_\infty^{\chi(X)} \cdot W_{1,2}$ in our theorem, it remains to study $C_{2,N}^\mathfrak{T}$. For $1 \leq i \leq N$, let $\ell_{i,+}$ (resp. $\ell_{i,-}$) be the sum of the multiplicities of the positive (resp. negative) parts in $\mu^{(i)}$. Denote the parts (repeated with multiplicities) of $\mu^{(i)}$ by $-n'_{j_{i,1}}, \dots, -n'_{j_{i,\ell_{i,-}}}, n'_{h_{i,1}}, \dots, n'_{h_{i,\ell_{i,+}}}$. By the definition of \mathfrak{T} , the following data are the same for every N -tuple $(\tilde{\mu}^{(1)}, \dots, \tilde{\mu}^{(N)}) \in \mathfrak{T}$:

- the indexes $j_{i,1}, \dots, j_{i,\ell_{i,-}}$ ($1 \leq i \leq N$) up to re-ordering
- the indexes $h_{i,1}, \dots, h_{i,\ell_{i,+}}$ ($1 \leq i \leq N$) up to re-ordering
- the partition $\{\pi_1, \dots, \pi_u\}$ of $\{1, \dots, N\}$ and integers w_i, p_i in (4.30)
- the coefficient of line (4.30) in the linear combination.

So by (4.28), (4.29) and (4.30), $C_{2,N}^\mathfrak{T}$ is a linear combination of the expressions

$$(q; q)_\infty^{-\chi(X)} \sum_{\substack{n_1, \dots, n_\ell > 0 \\ n_i \neq n_j \text{ if } i \neq j}} \prod_{i=1}^\ell \frac{(-n_i)^{w_i} q^{n_i p_i}}{(1 - q^{n_i})^{w_i}} \sum_{\substack{\ell(\tilde{\lambda}^{(i)}) = k_i + 2 - \ell_{i,+} - \ell_{i,-} \\ |\tilde{\lambda}^{(i)}| = \sum_{r=1}^{\ell_{i,-}} n_{j_{i,r}} - \sum_{r=1}^{\ell_{i,+}} n_{h_{i,r}} \\ 1 \leq i \leq N}} \prod_{i=1}^u \left\langle 1_X, \prod_{j \in \pi_i} ((1_X - K_X)^{\sum_{n \geq 1} p_n^{(j)}} \alpha_j) \right\rangle \cdot \prod_{\substack{1 \leq i \leq N \\ n \geq 1}} \left(\frac{(-1)^{p_n^{(i)}}}{p_n^{(i)}!} \frac{q^{n p_n^{(i)}}}{(1 - q^n)^{p_n^{(i)}}} \frac{1}{\tilde{p}_n^{(i)}!} \frac{1}{(1 - q^n)^{\tilde{p}_n^{(i)}}} \right)$$

(we have moved the factor $(-1)^N$ into the coefficients of the linear combination). Inserting the variables z_1, \dots, z_N , we conclude that $(q; q)_\infty^{\chi(X)} \cdot C_{2,N}^\mathfrak{T}$ is a linear combination of the coefficients of $z_1^0 \cdots z_N^0$ in the expressions

$$\left(\sum_{\substack{n_1, \dots, n_\ell > 0 \\ n_i \neq n_j \text{ if } i \neq j}} \prod_{i=1}^\ell \frac{(-n_i)^{w_i} q^{n_i p_i}}{(1 - q^{n_i})^{w_i}} \cdot \prod_{1 \leq i \leq N} z_i^{-\sum_{r=1}^{\ell_{i,-}} n_{j_{i,r}} + \sum_{r=1}^{\ell_{i,+}} n_{h_{i,r}}} \right) \quad (4.31)$$

$$\cdot \sum_{\substack{\ell(\vec{\lambda}^{(i)})=k_i+2-\ell_{i,+}-\ell_{i,-} \\ 1 \leq i \leq N}} \prod_{i=1}^u \left\langle 1_X, \prod_{j \in \pi_i} ((1_X - K_X)^{\sum_{n \geq 1} p_n^{(j)}} \alpha_j) \right\rangle \quad (4.32)$$

$$\cdot \prod_{\substack{1 \leq i \leq N \\ n \geq 1}} \left(\frac{(-1)^{p_n^{(i)}}}{p_n^{(i)}!} \frac{(qz_i)^{np_n^{(i)}}}{(1-q^n)^{p_n^{(i)}}} \frac{1}{\tilde{p}_n^{(i)}!} \frac{z_i^{-n\tilde{p}_n^{(i)}}}{(1-q^n)^{\tilde{p}_n^{(i)}}} \right). \quad (4.33)$$

We claim that line (4.31) is the sum of $\ell!$ multiple q -zeta values of weight $\sum_{i=1}^{\ell} w_i$. Indeed, the sum of the terms with $n_1 > \dots > n_{\ell}$ in line (4.31) is equal to:

$$\begin{aligned} & \sum_{n_1 > \dots > n_{\ell}} \prod_{i=1}^{\ell} \frac{(-n_i)^{w_i} q^{n_i p_i}}{(1-q^{n_i})^{w_i}} \cdot \prod_{1 \leq i \leq N} z_i^{-\sum_{r=1}^{\ell_{i,-}} n_{j_{i,r}} + \sum_{r=1}^{\ell_{i,+}} n_{h_{i,r}}} \\ &= \sum_{n_1 > \dots > n_{\ell}} \prod_{i=1}^{\ell} \frac{(-n_i)^{w_i} q^{n_i p_i} f_i(z_1, \dots, z_N)^{n_i}}{(1-q^{n_i})^{w_i}} \end{aligned} \quad (4.34)$$

where each $f_i(z_1, \dots, z_N)$ is a suitable monomial of $z_1^{\pm 1}, \dots, z_N^{\pm 1}$. So line (4.31) is the sum of the following $\ell!$ multiple q -zeta values:

$$\sum_{n_1 > \dots > n_{\ell}} \prod_{i=1}^{\ell} \frac{(-n_i)^{w_{\sigma(i)}} q^{n_i p_{\sigma(i)}} f_{\sigma(i)}(z_1, \dots, z_N)^{n_i}}{(1-q^{n_i})^{w_{\sigma(i)}}} \quad (4.35)$$

where σ runs in the symmetric group S_{ℓ} . Furthermore, as in the proof of Lemma 4.7, the product of lines (4.32) and (4.33) is equal to

$$\begin{aligned} & \sum_{\substack{a_i, b_i \geq 0; s_1^{(i)}, \dots, s_{a_i}^{(i)}, t_1^{(i)}, \dots, t_{b_i}^{(i)} \geq 1 \\ \sum_{r=1}^{a_i} s_r^{(i)} + \sum_{r=1}^{b_i} t_r^{(i)} = k_i + 2 - \ell_{i,+} - \ell_{i,-} \\ 1 \leq i \leq N}} \prod_{i=1}^u \left\langle 1_X, \prod_{j \in \pi_i} ((1_X - K_X)^{\sum_{r=1}^{a_j} s_r^{(j)}} \alpha_j) \right\rangle \cdot \prod_{\substack{1 \leq r \leq a_i \\ 1 \leq i \leq N}} \frac{(-1)^{s_r^{(i)}}}{s_r^{(i)}!} \\ & \cdot \prod_{\substack{1 \leq r \leq b_i \\ 1 \leq i \leq N}} \frac{1}{t_r^{(i)}!} \cdot \prod_{i=1}^N \left(\sum_{n_1 > \dots > n_{a_i}} \prod_{r=1}^{a_i} \frac{(qz_i)^{n_r s_r^{(i)}}}{(1-q^{n_r})^{s_r^{(i)}}} \cdot \sum_{m_1 > \dots > m_{b_i}} \prod_{r=1}^{b_i} \frac{z_i^{-m_r t_r^{(i)}}}{(1-q^{m_r})^{t_r^{(i)}}} \right). \end{aligned}$$

Combining with lines (4.31) and (4.35), we see that $(q; q)_{\infty}^{\chi(X)} \cdot C_{2,N}^{\vec{\gamma}}$ is a linear combination of the coefficients of $z_1^0 \dots z_N^0$ in some multiple q -zeta values of weights

$$\begin{aligned} w &:= \sum_{i=1}^{\ell} w_i + \sum_{i=1}^N \left(\sum_{r=1}^{a_i} s_r^{(i)} + \sum_{r=1}^{b_i} t_r^{(i)} \right) \\ &= \sum_{i=1}^{\ell} w_i + \sum_{i=1}^N (k_i + 2 - \ell_{i,+} - \ell_{i,-}). \end{aligned}$$

Note from (4.30) that $\sum_{i=1}^{\ell} w_i = \sum_{i=1}^N \ell(\mu^{(i)})/2 = \sum_{i=1}^N (\ell_{i,+} + \ell_{i,-})/2$. So we have $w < \sum_{i=1}^N (k_i + 2)$. This completes the proof of our theorem. \square

We will end this section with three propositions about $F_k^\alpha(q)$, which provide some insight into the lower weight term W_1 in Theorem 4.8. Proposition 4.11 deals with $F_0^\alpha(q)$ for an arbitrary $\alpha \in H^*(X)$. Proposition 4.13 calculates $F_1^\alpha(q)$ by assuming $e_X \alpha = 0$. Proposition 4.14 computes $F_k^\alpha(q)$, $k \geq 2$ by assuming $e_X \alpha = K_X \alpha = 0$.

Proposition 4.11. *The generating series $F_0^\alpha(q)$ is equal to*

$$(q; q)_\infty^{-\chi(X)} \cdot \langle 1_X - K_X, \alpha \rangle \cdot \sum_n \frac{q^n}{(1 - q^n)^2} + (q; q)_\infty^{-\chi(X)} \cdot \langle e_X, \alpha \rangle \cdot \sum_n \frac{nq^n}{1 - q^n}. \quad (4.36)$$

Proof. By Lemma 3.2, $F_k^\alpha(q) = \text{Tr } q^\partial W(\mathfrak{L}_1, z) \mathfrak{G}_k(\alpha)$. By Theorem 2.3, we have $\mathfrak{G}_0(\alpha) = -\sum_{n>0} (\mathfrak{a}_{-n} \mathfrak{a}_n)(\alpha)$. Now (4.36) follows from Remark 4.6. \square

Remark 4.12. By (4.36), $F_0^{1_X}(q) = (q; q)_\infty^{-\chi(X)} \cdot \chi(X) \cdot \sum_n \frac{nq^n}{1 - q^n} = q \frac{d}{dq} (q; q)_\infty^{-\chi(X)}$.

Proposition 4.13. *Let $\alpha \in H^*(X)$ be a homogeneous class satisfying $e_X \alpha = 0$. Then, the generating series $F_1^\alpha(q)$ is the coefficient of z^0 in*

$$(q; q)_\infty^{-\chi(X)} \cdot \frac{\langle K_X - K_X^2, \alpha \rangle}{2} \cdot \left(\sum_n \frac{(n-1)q^n}{(1 - q^n)^2} + \sum_n \frac{(qz)^n}{1 - q^n} \cdot \left(\sum_m \frac{z^{-2m}}{(1 - q^m)^2} + 2 \sum_{m_1 > m_2} \frac{z^{-m_1}}{1 - q^{m_1}} \frac{z^{-m_2}}{1 - q^{m_2}} \right) \right).$$

Proof. We have $F_1^\alpha(q) = \text{Tr } q^\partial W(\mathfrak{L}_1, z) \mathfrak{G}_1(\alpha)$. It is known that

$$\mathfrak{G}_1(\alpha) = - \sum_{\ell(\lambda)=3, |\lambda|=0} \frac{\mathfrak{a}_\lambda(\alpha)}{\lambda!} - \sum_{n>0} \frac{n-1}{2} (\mathfrak{a}_{-n} \mathfrak{a}_n)(K_X \alpha). \quad (4.37)$$

Applying Remark 4.6 to $-\sum_{n>0} \frac{n-1}{2} \text{Tr } q^\partial W(\mathfrak{L}_1, z) (\mathfrak{a}_{-n} \mathfrak{a}_n)(K_X \alpha)$ yields the weight-

2 terms in our proposition. Again by Remark 4.6, the trace $\text{Tr } q^\partial W(\mathfrak{L}_1, z) \frac{\mathfrak{a}_\lambda(\alpha)}{\lambda!}$ with $\ell(\lambda) = 3$ and $|\lambda| = 0$ contains only weight-3 terms (i.e., does not contain lower weight terms). So the proof of Theorem 4.8 shows that

$$- \sum_{\ell(\lambda)=3, |\lambda|=0} \text{Tr } q^\partial W(\mathfrak{L}_1, z) \frac{\mathfrak{a}_\lambda(\alpha)}{\lambda!} = (q; q)_\infty^{-\chi(X)} \cdot \text{Coeff}_{z^0} \Theta_1^\alpha(q, z).$$

Expanding $\text{Coeff}_{z^0} \Theta_1^\alpha(q, z)$ yields the weight-3 terms in our proposition. \square

Proposition 4.14. *Let $\alpha \in H^*(X)$ be homogeneous satisfying $K_X \alpha = e_X \alpha = 0$.*

- (i) *If $|\alpha| < 4$, then $F_k^\alpha(q) = 0$ for every $k \geq 0$;*
- (ii) *Let $|\alpha| = 4$ and $k \geq 0$. Then, $F_k^\alpha(q)$ is the coefficient of z^0 in*

$$-(q; q)_\infty^{-\chi(X)} \cdot \langle 1_X, \alpha \rangle \cdot \sum_{\substack{a, s_1, \dots, s_a, b, t_1, \dots, t_b \geq 1 \\ \sum_{i=1}^a s_i + \sum_{j=1}^b t_j = k+2}} \prod_{i=1}^a \frac{(-1)^{s_i}}{s_i!} \cdot \prod_{j=1}^b \frac{1}{t_j!}$$

$$\cdot \sum_{n_1 > \dots > n_a} \prod_{i=1}^a \frac{(qz)^{n_i s_i}}{(1 - q^{n_i})^{s_i}} \cdot \sum_{m_1 > \dots > m_b} \prod_{j=1}^b \frac{z^{-m_j t_j}}{(1 - q^{m_j})^{t_j}}. \quad (4.38)$$

In particular, if $2 \nmid k$, then $F_k^\alpha(q) = 0$.

Proof. Since $K_X \alpha = e_X \alpha = 0$, we conclude from Theorem 2.3 that

$$\mathfrak{G}_k(\alpha) = - \sum_{\ell(\lambda)=k+2, |\lambda|=0} \frac{\mathfrak{a}_\lambda(\alpha)}{\lambda!}.$$

As in the proof of Proposition 4.13, Remark 4.6 and the proof of Theorem 4.8 yield

$$F_k^\alpha(q) = (q; q)_\infty^{-\chi(X)} \cdot \text{Coeff}_{z^0} \Theta_k^\alpha(q, z).$$

By the definition of $\Theta_k^\alpha(q, z)$ in (4.17), we see that (i) holds and that our formula for $F_k^\alpha(q)$ with $|\alpha| = 4$ and $k \geq 0$ holds. Note that line (4.38) can be rewritten as

$$\sum_{n_1 > \dots > n_a} \prod_{i=1}^a \frac{(qz^2)^{n_i s_i / 2}}{(1 - q^{n_i})^{s_i}} \cdot \sum_{m_1 > \dots > m_b} \prod_{j=1}^b \frac{(qz^{-2})^{m_j t_j / 2}}{(1 - q^{m_j})^{t_j}}.$$

Therefore, if $|\alpha| = 4$ and $2 \nmid k$, then the role of a, s_1, \dots, s_a and the role of b, t_1, \dots, t_b in the above formula of $F_k^\alpha(q)$ are anti-symmetric; so $F_k^\alpha(q) = 0$. \square

5. The reduced series $\langle \text{ch}_{k_1}^{L_1} \cdots \text{ch}_{k_N}^{L_N} \rangle'$

In this section, we will prove Conjecture 1.1 modulo the lower weight term. Moreover, for abelian surfaces, we will verify Conjecture 1.1.

Let L be a line bundle on the smooth projective surface X . It induces the tautological rank- n bundle $L^{[n]}$ over the Hilbert scheme $X^{[n]}$:

$$L^{[n]} = p_{1*}(p_2^* L|_{\mathcal{Z}_n})$$

where \mathcal{Z}_n is the universal codimension-2 subscheme of $X^{[n]} \times X$, and p_1 and p_2 are the projections of $X^{[n]} \times X$ to $X^{[n]}$ and X respectively. By the Grothendieck-Riemann-Roch Theorem and (2.1), we obtain

$$\begin{aligned} \text{ch}(L^{[n]}) &= p_{1*}(\text{ch}(\mathcal{O}_{\mathcal{Z}_n}) \cdot p_2^* \text{ch}(L) \cdot p_2^* \text{td}(X)) \\ &= G(1_X, n) + G(L, n) + G(L^2/2, n). \end{aligned} \quad (5.1)$$

Since the cohomology degree of $G_i(\alpha, n)$ is $2i + |\alpha|$, we have

$$\text{ch}_k(L^{[n]}) = G_k(1_X, n) + G_{k-1}(L, n) + G_{k-2}(L^2/2, n). \quad (5.2)$$

Following Okounkov [Ok], we have defined the generating series $\langle \text{ch}_{k_1}^{L_1} \cdots \text{ch}_{k_N}^{L_N} \rangle$ and its reduced version $\langle \text{ch}_{k_1}^{L_1} \cdots \text{ch}_{k_N}^{L_N} \rangle'$ in (1.1) and (1.2) respectively.

Theorem 5.1. *Let L_1, \dots, L_N be line bundles over X , and $k_1, \dots, k_N \geq 0$. Then,*

$$\langle \text{ch}_{k_1}^{L_1} \cdots \text{ch}_{k_N}^{L_N} \rangle' = \text{Coeff}_{z_1^0 \cdots z_N^0} \left(\prod_{i=1}^N \Theta_{k_i}^{1_X}(q, z_i) \right) + W, \quad (5.3)$$

and the lower weight term W is an infinite linear combination of the expressions:

$$\prod_{i=1}^u \left\langle K_X^{r_i} e_X^{r'_i}, L_1^{\ell_{i,1}} \cdots L_N^{\ell_{i,N}} \right\rangle \cdot \prod_{i=1}^v \frac{q^{n_i w_i p_i}}{(1 - q^{n_i})^{w_i}}$$

where $\sum_{i=1}^v w_i < \sum_{i=1}^N (k_i + 2)$, and the integers $u, v, r_i, r'_i, \ell_{i,j} \geq 0, n_i > 0, w_i > 0, p_i \in \{0, 1\}$ depend only on k_1, \dots, k_N . Furthermore, all the coefficients of this linear combination are independent of q, L_1, \dots, L_N and X .

Proof. We conclude from (1.1), (1.2), (5.2) and (2.2) that

$$\langle \text{ch}_{k_1}^L \cdots \text{ch}_{k_N}^L \rangle' = (q; q)_\infty^{\chi(X)} \cdot F_{k_1, \dots, k_N}^{1_X, \dots, 1_X}(q) + (q; q)_\infty^{\chi(X)} \cdot A$$

where A is the sum of the series $F_{k'_1, \dots, k'_N}^{\alpha_1, \dots, \alpha_N}$ such that for every $1 \leq i \leq N$,

$$(\alpha_i, k'_i) \in \{(1_X, k_i), (L_i, k_i - 1), (L_i^2/2, k_i - 2)\},$$

and $\sum_{i=1}^N k'_i < \sum_{i=1}^N k_i$. Now our result follows from Theorem 4.8. \square

Theorem 5.2. *Let L_1, \dots, L_N be line bundles over an abelian surface X , and $k_1, \dots, k_N \geq 0$. Then, the lower weight term W in (5.3) is a linear combination of the coefficients of $z_1^0 \cdots z_N^0$ in some multiple q -zeta values (with additional variables z_1, \dots, z_N inserted) of weights $< \sum_{i=1}^N (k_i + 2)$. Moreover, the coefficients in this linear combination are independent of q .*

Proof. Since $e_X = K_X = 0$, $F_{\tilde{k}_1, \dots, \tilde{k}_N}^{\tilde{\alpha}_1, \dots, \tilde{\alpha}_N} = \tilde{F}_{\tilde{k}_1, \dots, \tilde{k}_N}^{\tilde{\alpha}_1, \dots, \tilde{\alpha}_N}$ by Lemma 3.2, Theorem 2.3 and (4.22). By Theorem 4.10 and the proof of Theorem 5.1, our theorem follows. \square

Our next two propositions compute the series $\langle \text{ch}_k^L \rangle$ completely, and should offer some insight into the lower weight term W in Theorem 5.1. Proposition 5.3 calculates $\langle \text{ch}_1^L \rangle$ by assuming $e_X = 0$, while Proposition 5.4 deals with the series $\langle \text{ch}_k^L \rangle, k \geq 2$ by assuming $e_X = K_X = 0$ (i.e., by assuming that X is an abelian surface). Note from (5.2) that when $\chi(X) = 0$, we have

$$\langle \text{ch}_k^L \rangle' = \langle \text{ch}_k^L \rangle = F_k^{1_X}(q) + F_{k-1}^L(q) + \frac{1}{2} \cdot F_{k-2}^{L^2}(q). \quad (5.4)$$

Proposition 5.3. *Let L be a line bundle over a smooth projective surface X with $e_X = 0$. Then, the series $\langle \text{ch}_1^L \rangle$ is the coefficient of z^0 in*

$$\begin{aligned} & -\langle K_X, L \rangle \cdot \sum_n \frac{q^n}{(1 - q^n)^2} - \frac{\langle K_X, K_X \rangle}{2} \cdot \sum_n \frac{(n-1)q^n}{(1 - q^n)^2} \\ & - \frac{\langle K_X, K_X \rangle}{2} \cdot \sum_n \frac{(qz)^n}{1 - q^n} \cdot \left(\sum_m \frac{z^{-2m}}{(1 - q^m)^2} + 2 \sum_{m_1 > m_2} \frac{z^{-m_1}}{1 - q^{m_1}} \frac{z^{-m_2}}{1 - q^{m_2}} \right). \end{aligned}$$

Proof. Our formula follows from (5.4), (4.36) and Proposition 4.13. \square

Proposition 5.4. *Let L be a line bundle over an abelian surface X . If $2 \nmid k$, then $\langle \text{ch}_k^L \rangle' = 0$. If $2 \mid k$, the generating series $\langle \text{ch}_k^L \rangle'$ is the coefficient of z^0 in*

$$-\frac{\langle L, L \rangle}{2} \cdot \sum_{\substack{a, s_1, \dots, s_a, b, t_1, \dots, t_b \geq 1 \\ \sum_{i=1}^a s_i + \sum_{j=1}^b t_j = k}} \prod_{i=1}^a \frac{(-1)^{s_i}}{s_i!} \cdot \prod_{j=1}^b \frac{1}{t_j!} \\ \cdot \sum_{n_1 > \dots > n_a} \prod_{i=1}^a \frac{(qz)^{n_i s_i}}{(1 - q^{n_i})^{s_i}} \cdot \sum_{m_1 > \dots > m_b} \prod_{j=1}^b \frac{z^{-m_j t_j}}{(1 - q^{m_j})^{t_j}}.$$

Proof. Follows immediately from (5.4) and Proposition 4.14. \square

6. Applications to the universal constants in $\sum_n c(T_{X^{[n]}}) q^n$

Let $x \in H^4(X)$ be the cohomology class of a point in the surface X . In this section, we will compute $F_{k_1, \dots, k_N}^{x, \dots, x}(q)$ in terms of the universal constants in the expression of $\sum_n c(T_{X^{[n]}}) q^n$ formulated in [Boi, BN]. Comparing with Proposition 4.14 (ii) enables us to determine some of these universal constants.

Let $C_i = \binom{2i}{i}/(i+1)$ be the Catalan number and $\sigma_1(i) = \sum_{j \mid i} j$. Recall that $\mathcal{P} = \tilde{\mathcal{P}}_+$ is the set of all the usual partitions. The following lemma is from [Boi, BN].

Lemma 6.1. *There exist unique rational numbers $b_\mu, f_\mu, g_\mu, h_\mu$ depending only on the partitions $\mu \in \mathcal{P}$ such that $\sum_n c(T_{X^{[n]}}) q^n$ is equal to*

$$\exp \left(\sum_{\mu \in \mathcal{P}} q^{|\mu|} \left(b_\mu \mathbf{a}_{-\mu}(1_X) + f_\mu \mathbf{a}_{-\mu}(e_X) + g_\mu \mathbf{a}_{-\mu}(K_X) + h_\mu \mathbf{a}_{-\mu}(K_X^2) \right) \right) |0\rangle.$$

In addition, for $i \geq 1$, we have $b_{2i} = 0$, $b_{2i-1} = (-1)^{i-1} C_{i-1}/(2i-1)$, $b_{(1^i)} = f_{(1^i)} = -g_{(1^i)} = \sigma_1(i)/i$, and $h_{(1^i)} = 0$.

Our goal is to compute $F_{k_1, \dots, k_N}^{\alpha_1, \dots, \alpha_N}(q)$ in terms of the universal constants b_μ, f_μ, g_μ and h_μ . Using the definition of the operators $\mathfrak{G}_{k_i}(\alpha_i)$, we see that

$$\begin{aligned} F_{k_1, \dots, k_N}^{\alpha_1, \dots, \alpha_N}(q) &= \sum_n q^n \left\langle \left(\prod_{i=1}^N G_{k_i}(\alpha_i, n) \right) c(T_{X^{[n]}}), 1_{X^{[n]}} \right\rangle \\ &= \sum_n q^n \left\langle \left(\prod_{i=1}^N \mathfrak{G}_{k_i}(\alpha_i) \right) c(T_{X^{[n]}}), 1_{X^{[n]}} \right\rangle \\ &= \left\langle \left(\prod_{i=1}^N \mathfrak{G}_{k_i}(\alpha_i) \right) \sum_n c(T_{X^{[n]}}) q^n, |1\rangle \right\rangle \end{aligned} \quad (6.1)$$

where we have put $|1\rangle = \sum_n 1_{X^{[n]}} = \exp(\mathbf{a}_{-1}(1_X)) \cdot |0\rangle$.

Lemma 6.2. *Let $w \in \mathbb{H}_X$, and \mathfrak{G} be a (possibly infinite) sum of monomials of Heisenberg creation operators. Then, $\langle \mathfrak{G}w, |1\rangle \rangle = \langle \mathfrak{G}|0\rangle, |1\rangle \rangle \cdot \langle w, |1\rangle \rangle$.*

Proof. By linearity, it suffices to prove that

$$\left\langle \prod_{i=1}^s \mathfrak{a}_{-n_i}(\alpha_i) \cdot \prod_{j=1}^t \mathfrak{a}_{-m_j}(\beta_j) |0\rangle, |1\rangle \right\rangle = \left\langle \prod_{i=1}^s \mathfrak{a}_{-n_i}(\alpha_i) |0\rangle, |1\rangle \right\rangle \cdot \left\langle \prod_{j=1}^t \mathfrak{a}_{-m_j}(\beta_j) |0\rangle, |1\rangle \right\rangle$$

where $n_1, \dots, n_s, m_1, \dots, m_t > 0$, and $\alpha_1, \dots, \alpha_s, \beta_1, \dots, \beta_t$ are homogeneous. Indeed, if $\mathfrak{a}_{-n_i}(\alpha_i) \notin \mathbb{C} \mathfrak{a}_{-1}(x)$ for some i or if $\mathfrak{a}_{-m_j}(\beta_j) \notin \mathbb{C} \mathfrak{a}_{-1}(x)$ for some j , then both sides are equal to 0. Otherwise, letting $\mathfrak{a}_{-n_i}(\alpha_i) = u_i \mathfrak{a}_{-1}(x)$ for every i and $\mathfrak{a}_{-m_j}(\beta_j) = v_j \mathfrak{a}_{-1}(x)$ for every j , we see that both sides are $u_1 \cdots u_s v_1 \cdots v_t$. \square

Lemma 6.3. *Let $b_{(1j)}$ and $b_{(i,1j)}$ be from Lemma 6.1. Let $\tilde{b}_{(i,1j)} = (j+1)b_{(1j+1)}$ if $i = 1$, and $\tilde{b}_{(i,1j)} = b_{(i,1j)}$ if $i > 1$. Then, $F_{k_1, \dots, k_N}^{x, \dots, x}(q)$ is equal to*

$$(q; q)_{\infty}^{-\chi(X)} \cdot (-1)^N \sum_{\substack{\sum_{s \geq 1} (s+1)m_{i,s} = k_i + 2 \\ 1 \leq i \leq N}} \prod_{i=1}^N \frac{1}{(\sum_{s \geq 1} s m_{i,s})!} \cdot \prod_{s \geq 1} \left(\frac{(-s)^{m_s} m_s!}{\prod_{i=1}^N m_{i,s}!} \sum_{t_0 + t_1 + \dots + t_j + \dots = m_s} \prod_{j=0}^{+\infty} \frac{(\tilde{b}_{(s,1j)} q^{s+j})^{t_j}}{t_j!} \right)$$

where $m_{i,s} \geq 0$ for every i and s , and $m_s = \sum_{i=1}^N m_{i,s}$ for every $s \geq 1$.

Proof. By (6.1) and Theorem 2.3, we obtain

$$F_{k_1, \dots, k_N}^{x, \dots, x}(q) = \left\langle \left(\prod_{i=1}^N \mathfrak{G}_{k_i}(x) \right) \sum_n c(T_{X^{[n]}}) q^n, |1\rangle \right\rangle, \quad (6.2)$$

$$\prod_{i=1}^N \mathfrak{G}_{k_i}(x) = (-1)^N \sum_{\substack{\ell(\lambda^{(i)}) = k_i + 2, |\lambda^{(i)}| = 0 \\ 1 \leq i \leq N}} \prod_{i=1}^N \frac{\mathfrak{a}_{\lambda^{(i)}}(x)}{(\lambda^{(i)})!}. \quad (6.3)$$

Note that $\tau_{\ell*} x = \underbrace{x \otimes \cdots \otimes x}_{\ell \text{ times}}$, $K_X^2 = \langle K_X, K_X \rangle x$, $e_X = \chi(X)x$, and

$$\tau_{\ell*} 1_X = 1_X \otimes \underbrace{x \otimes \cdots \otimes x}_{(\ell-1) \text{ times}} + \cdots + \underbrace{x \otimes \cdots \otimes x}_{(\ell-1) \text{ times}} \otimes 1_X + w \quad (6.4)$$

where w is a sum of cohomology classes of the form $\alpha_1 \otimes \cdots \otimes \alpha_{\ell}$ with $0 < |\alpha_i| < 4$ for some i . So for a generalized partition λ , positive integers n_1, \dots, n_s and homogeneous classes $\alpha_1, \dots, \alpha_s \in H^*(X)$, we have

$$\langle \mathfrak{a}_{\lambda}(x) \mathfrak{a}_{-n_1}(\alpha_1) \cdots \mathfrak{a}_{-n_s}(\alpha_s) |0\rangle, |1\rangle \rangle = 0 \quad (6.5)$$

if $0 < |\alpha_i| < 4$ for some i , or if $\mathbf{a}_{-n_i}(\alpha_i) \in \mathbb{C} \mathbf{a}_{-j}(x)$ for some i and for some $j > 1$. Combining with (6.2), (6.3) and Lemma 6.1, we see that $F_{k_1, \dots, k_N}^{x, \dots, x}(q)$ equals

$$\begin{aligned} & \left\langle \left(\prod_{i=1}^N \mathfrak{G}_{k_i}(x) \right) \exp \left(\sum_{\mu \in \mathcal{P}} b_\mu \mathbf{a}_{-\mu}(1_X) q^{|\mu|} + \sum_i \tilde{f}_{(1^i)} \mathbf{a}_{-(1^i)}(x) q^i \right) |0\rangle, |1\rangle \right\rangle \\ &= \left\langle \exp \left(\sum_i \tilde{f}_{(1^i)} \mathbf{a}_{-(1^i)}(x) q^i \right) \cdot \prod_{i=1}^N \mathfrak{G}_{k_i}(x) \cdot \exp \left(\sum_{\mu \in \mathcal{P}} b_\mu \mathbf{a}_{-\mu}(1_X) q^{|\mu|} \right) |0\rangle, |1\rangle \right\rangle \end{aligned}$$

where $\tilde{f}_{(1^i)} = \chi(X) \cdot f_{(1^i)}$. By Lemma 6.2, $F_{k_1, \dots, k_N}^{x, \dots, x}(q)$ is equal to

$$\begin{aligned} & \left\langle \exp \left(\sum_i \tilde{f}_{(1^i)} \mathbf{a}_{-(1^i)}(x) q^i \right) |0\rangle, |1\rangle \right\rangle \\ & \cdot \left\langle \prod_{i=1}^N \mathfrak{G}_{k_i}(x) \cdot \exp \left(\sum_{\mu \in \mathcal{P}} b_\mu \mathbf{a}_{-\mu}(1_X) q^{|\mu|} \right) |0\rangle, |1\rangle \right\rangle. \end{aligned} \quad (6.6)$$

In particular, setting $N = 0$, we conclude that

$$\left\langle \exp \left(\sum_i \tilde{f}_{(1^i)} \mathbf{a}_{-(1^i)}(x) q^i \right) |0\rangle, |1\rangle \right\rangle = F(q) = (q; q)_\infty^{-\chi(X)}. \quad (6.7)$$

It follows from (6.6) and (6.4) that $F_{k_1, \dots, k_N}^{x, \dots, x}(q)$ is equal to

$$\begin{aligned} & (q; q)_\infty^{-\chi(X)} \left\langle \prod_{i=1}^N \mathfrak{G}_{k_i}(x) \cdot \exp \left(\sum_{\mu \in \mathcal{P}} b_\mu \mathbf{a}_{-\mu}(1_X) q^{|\mu|} \right) |0\rangle, |1\rangle \right\rangle \\ &= (q; q)_\infty^{-\chi(X)} \left\langle \prod_{i=1}^N \mathfrak{G}_{k_i}(x) \cdot \exp \left(\sum_{\substack{i \geq 1 \\ j \geq 0}} \tilde{b}_{(i, 1^j)} \mathbf{a}_{-i}(1_X) \mathbf{a}_{-1}(x)^j q^{i+j} \right) |0\rangle, |1\rangle \right\rangle \end{aligned} \quad (6.8)$$

where $\tilde{b}_{(i, 1^j)} = (j+1)b_{(1^{j+1})}$ if $i = 1$, and $\tilde{b}_{(i, 1^j)} = b_{(i, 1^j)}$ if $i > 1$. Let $\lambda^{(1)}, \dots, \lambda^{(N)}$ be from the right-hand-side of (6.3). In order to have a nonzero pairing

$$\left\langle \prod_{i=1}^N \mathbf{a}_{\lambda^{(i)}}(x) \cdot \exp \left(\sum_{i \geq 1, j \geq 0} \tilde{b}_{(i, 1^j)} \mathbf{a}_{-i}(1_X) \mathbf{a}_{-1}(x)^j q^{i+j} \right) |0\rangle, |1\rangle \right\rangle, \quad (6.9)$$

each $\lambda^{(i)}$ with $1 \leq i \leq N$ must be of the form $((-1)^{n_i} 1^{m_{i,1}} 2^{m_{i,2}} \dots)$; since $\ell(\lambda^{(i)}) = k_i + 2$ and $|\lambda^{(i)}| = 0$, we get $n_i + \sum_{s \geq 1} m_{i,s} = k_i + 2$ and $n_i = \sum_{s \geq 1} s m_{i,s}$; so

$$\sum_{s \geq 1} (s+1) m_{i,s} = k_i + 2. \quad (6.10)$$

In this case, using Lemma 6.2, we see that (6.9) is equal to

$$\begin{aligned} & \left\langle \mathbf{a}_{-1}(x)^{\sum_i n_i} \cdot \prod_{i,s} \mathbf{a}_s(x)^{m_{i,s}} \cdot \exp \left(\sum_{i \geq 1, j \geq 0} \tilde{b}_{(i,1^j)} \mathbf{a}_{-i}(1_X) \mathbf{a}_{-1}(x)^j q^{i+j} \right) |0\rangle, |1\rangle \right\rangle \\ &= \left\langle \prod_{1 \leq i \leq N, s \geq 1} \mathbf{a}_s(x)^{m_{i,s}} \cdot \exp \left(\sum_{i \geq 1, j \geq 0} \tilde{b}_{(i,1^j)} \mathbf{a}_{-i}(1_X) \mathbf{a}_{-1}(x)^j q^{i+j} \right) |0\rangle, |1\rangle \right\rangle. \end{aligned}$$

Put $m_s = \sum_{i=1}^N m_{i,s}$ for every $s \geq 1$. Then, (6.9) is equal to

$$\begin{aligned} & \left\langle \prod_{s \geq 1} \mathbf{a}_s(x)^{m_s} \cdot \exp \left(\sum_{i \geq 1, j \geq 0} \tilde{b}_{(i,1^j)} \mathbf{a}_{-i}(1_X) \mathbf{a}_{-1}(x)^j q^{i+j} \right) |0\rangle, |1\rangle \right\rangle \\ &= \left\langle \prod_{s \geq 1} \mathbf{a}_s(x)^{m_s} \cdot \prod_{i \geq 1, j \geq 0} \sum_t \frac{1}{t!} \left(\tilde{b}_{(i,1^j)} \mathbf{a}_{-i}(1_X) \mathbf{a}_{-1}(x)^j q^{i+j} \right)^t \cdot |0\rangle, |1\rangle \right\rangle \\ &= \prod_{s \geq 1} \left((-s)^{m_s} m_s! \sum_{t_0+t_1+\dots+t_j+\dots=m_s} \prod_{j=0}^{+\infty} \frac{(\tilde{b}_{(s,1^j)} q^{s+j})^{t_j}}{t_j!} \right). \end{aligned}$$

Combining this with (6.8), (6.3), (6.9) and (6.10), $F_{k_1, \dots, k_N}^{x, \dots, x}(q)$ is equal to

$$\begin{aligned} & (q; q)_\infty^{-\chi(X)} \cdot (-1)^N \sum_{\substack{\sum_{s \geq 1} (s+1)m_{i,s} = k_i+2 \\ 1 \leq i \leq N}} \prod_{i=1}^N \frac{1}{(\sum_{s \geq 1} s m_{i,s})!} \\ & \cdot \prod_{s \geq 1} \left(\frac{(-s)^{m_s} m_s!}{\prod_i m_{i,s}!} \sum_{t_0+t_1+\dots+t_j+\dots=m_s} \prod_{j=0}^{+\infty} \frac{(\tilde{b}_{(s,1^j)} q^{s+j})^{t_j}}{t_j!} \right) \end{aligned}$$

where $m_{i,s} \geq 0$ for every i and s , and $m_s = \sum_{i=1}^N m_{i,s}$ for every $s \geq 1$. \square

Our next result determines the universal constants $b_{(i,1^j)}$ with $i \geq 2$ and $j \geq 0$.

Theorem 6.4. *Let the numbers $b_{(1^j)}$ and $b_{(i,1^j)}$ be from Lemma 6.1. Let $\tilde{b}_{(i,1^j)} = (j+1)b_{(1^{j+1})} = \sigma_1(j+1)$ if $i=1$, and $\tilde{b}_{(i,1^j)} = b_{(i,1^j)}$ if $i > 1$.*

(i) *If i is an even positive integer, then $b_{(i,1^j)} = 0$ for all $j \geq 0$.*

(ii) *Let $i > 1$ be odd. Then, $\frac{1}{(i-1)!} \sum_{j \geq 0} b_{(i,1^j)} q^{i+j}$ is equal to*

$$\begin{aligned} & \sum_{\substack{1 \leq s < i \\ 2 \nmid s}} \frac{1}{(\sum_{2 \nmid s} s m_s)!} \prod_{2 \nmid s} \left(\sum_{j \geq 0} \prod_{t_j=m_s}^{+\infty} \frac{((-s) \tilde{b}_{(s,1^j)} q^{s+j})^{t_j}}{t_j!} \right) \\ & - \sum_{\substack{a, s_1, \dots, s_a, b, t_1, \dots, t_b \geq 1 \\ \sum_{u=1}^a s_u + \sum_{v=1}^b t_v = i+1 \\ n_1 > \dots > n_a, m_1 > \dots > m_b \\ \sum_{u=1}^a n_u s_u = \sum_{v=1}^b m_v t_v}} \prod_{u=1}^a \frac{(-1)^{s_u} q^{n_u s_u}}{s_u! \cdot (1 - q^{n_u})^{s_u}} \cdot \prod_{v=1}^b \frac{1}{t_v! \cdot (1 - q^{m_v})^{t_v}}. \end{aligned}$$

Proof. (i) Setting $N = 1$ in Lemma 6.3, we see that $F_k^x(q)$ is equal to

$$-(q; q)_\infty^{-\chi(X)} \cdot \sum_{\sum_{s \geq 1} (s+1)m_s = k+2} \frac{1}{(\sum_{s \geq 1} sm_s)!} \prod_{s \geq 1} \left(\sum_{\sum_{j \geq 0} t_j = m_s} \prod_{j=0}^{+\infty} \frac{((-s)\tilde{b}_{(s,1^j)} q^{s+j})^{t_j}}{t_j!} \right).$$

Comparing this with (4.38) which holds for all $k \geq 0$, we obtain

$$\begin{aligned} & \sum_{\sum_{s \geq 1} (s+1)m_s = k+2} \frac{1}{(\sum_{s \geq 1} sm_s)!} \prod_{s \geq 1} \left(\sum_{\sum_{j \geq 0} t_j = m_s} \prod_{j=0}^{+\infty} \frac{((-s)\tilde{b}_{(s,1^j)} q^{s+j})^{t_j}}{t_j!} \right) \\ &= \sum_{\substack{a, s_1, \dots, s_a, b, t_1, \dots, t_b \geq 1 \\ \sum_{u=1}^a s_u + \sum_{v=1}^b t_v = k+2 \\ n_1 > \dots > n_a, m_1 > \dots > m_b \\ \sum_{u=1}^a n_u s_u = \sum_{v=1}^b m_v t_v}} \prod_{u=1}^a \frac{(-1)^{s_u} q^{n_u s_u}}{s_u! \cdot (1 - q^{n_u})^{s_u}} \cdot \prod_{v=1}^b \frac{1}{t_v! \cdot (1 - q^{m_v})^{t_v}}. \end{aligned}$$

The largest value of s satisfying $\sum_{s \geq 1} (s+1)m_s = k+2$ is given by $s = k+1$ together with $m_{k+1} = 1$. So the above identity can be rewritten as

$$\begin{aligned} & \frac{1}{k!} \sum_{j \geq 0} \tilde{b}_{(k+1, 1^j)} q^{(k+1)+j} \\ &= \sum_{\sum_{1 \leq s < k+1} (s+1)m_s = k+2} \frac{1}{(\sum_{s \geq 1} sm_s)!} \prod_{s \geq 1} \left(\sum_{\sum_{j \geq 0} t_j = m_s} \prod_{j=0}^{+\infty} \frac{((-s)\tilde{b}_{(s,1^j)} q^{s+j})^{t_j}}{t_j!} \right) \\ & \quad - \sum_{\substack{a, s_1, \dots, s_a, b, t_1, \dots, t_b \geq 1 \\ \sum_{u=1}^a s_u + \sum_{v=1}^b t_v = k+2 \\ n_1 > \dots > n_a, m_1 > \dots > m_b \\ \sum_{u=1}^a n_u s_u = \sum_{v=1}^b m_v t_v}} \prod_{u=1}^a \frac{(-1)^{s_u} q^{n_u s_u}}{s_u! \cdot (1 - q^{n_u})^{s_u}} \cdot \prod_{v=1}^b \frac{1}{t_v! \cdot (1 - q^{m_v})^{t_v}}. \end{aligned}$$

Replacing $k+1$ by i , we conclude that $\frac{1}{(i-1)!} \sum_{j \geq 0} \tilde{b}_{(i, 1^j)} q^{i+j}$ is equal to

$$\sum_{\sum_{1 \leq s < i} (s+1)m_s = i+1} \frac{1}{(\sum_{s \geq 1} sm_s)!} \prod_{s \geq 1} \left(\sum_{\sum_{j \geq 0} t_j = m_s} \prod_{j=0}^{+\infty} \frac{((-s)\tilde{b}_{(s,1^j)} q^{s+j})^{t_j}}{t_j!} \right) \quad (6.11)$$

$$- \sum_{\substack{a, s_1, \dots, s_a, b, t_1, \dots, t_b \geq 1 \\ \sum_{u=1}^a s_u + \sum_{v=1}^b t_v = i+1 \\ n_1 > \dots > n_a, m_1 > \dots > m_b \\ \sum_{u=1}^a n_u s_u = \sum_{v=1}^b m_v t_v}} \prod_{u=1}^a \frac{(-1)^{s_u} q^{n_u s_u}}{s_u! \cdot (1 - q^{n_u})^{s_u}} \cdot \prod_{v=1}^b \frac{1}{t_v! \cdot (1 - q^{m_v})^{t_v}}. \quad (6.12)$$

Note that (6.12) is equal to 0 if $2|i$. Letting $i = 2$, we get $\sum_{j \geq 0} \tilde{b}_{(2, 1^j)} q^{2+j} = 0$.

Therefore, $\tilde{b}_{(2, 1^j)} = 0$ for every $j \geq 0$. Hence we have $b_{(2, 1^j)} = 0$ for every $j \geq 0$.

Next, let $i > 2$ and $2|i$. Assume inductively that $b_{(s,1^j)} = 0$ for every $j \geq 0$ whenever $2 \leq s < i$ and $2|s$. Since (6.12) is 0, $\frac{1}{(i-1)!} \sum_{j \geq 0} b_{(i,1^j)} q^{i+j}$ is equal to

$$\sum_{\sum_{1 \leq s < i} (s+1)m_s = i+1} \frac{1}{(\sum_s s m_s)!} \prod_{s \geq 1} \left(\sum_{\sum_{j \geq 0} t_j = m_s} \prod_{j=0}^{+\infty} \frac{((-s)\tilde{b}_{(s,1^j)} q^{s+j})^{t_j}}{t_j!} \right)$$

The condition $\sum_{1 \leq s < i} (s+1)m_s = i+1$ implies that $m_s > 0$ for some even integer $s < i$. Hence $\frac{1}{(i-1)!} \sum_{j \geq 0} b_{(i,1^j)} q^{i+j} = 0$ by induction. So $b_{(i,1^j)} = 0$ for all $j \geq 0$.

(ii) Follows immediately from (i), (6.11) and (6.12). \square

Note that $b_{(2i)} = 0$, $i \geq 1$ has been proved in [Boi, BN] (see Lemma 6.1). Next, using the universal constants $f_{(2,1^j)}$, $g_{(2,1^j)}$ and $h_{(2,1^j)}$, we compute the generating series $F_1^\alpha(q)$ for a cohomology class α with $|\alpha| < 4$.

Lemma 6.5. *Let $f_{(2,1^j)}$, $g_{(2,1^j)}$ and $h_{(2,1^j)}$ be from Lemma 6.1, and let $\alpha \in H^*(X)$ be a homogeneous class with $0 < |\alpha| < 4$. Then,*

$$\begin{aligned} \text{(i)} \quad F_1^{1^X}(q) &= \sum_{j \geq 0} \tilde{f}_{(2,1^j)} q^{2+j} \text{ where } \tilde{f}_{(2,1^j)} = \chi(X) \cdot f_{(2,1^j)} + \langle K_X, K_X \rangle \cdot h_{(2,1^j)}; \\ \text{(ii)} \quad F_1^\alpha(q) &= (q; q)_\infty^{-\chi(X)} \cdot \sum_{j \geq 0} g_{(2,1^j)} q^{2+j} \cdot \langle \alpha, K_X \rangle. \end{aligned}$$

Proof. (i) Let $\alpha \in H^*(X)$ be an arbitrary cohomology class. Note that for all $n \geq 1$ and $A \in \mathbb{H}_X$, we have $\langle (n-1)(\mathbf{a}_{-n}\mathbf{a}_n)(K_X\alpha)A, |1\rangle \rangle = 0$. By (6.1) and (4.37),

$$\begin{aligned} F_1^\alpha(q) &= \left\langle \mathfrak{G}_1(\alpha) \sum_n c(T_{X^{[n]}}) q^n, |1\rangle \right\rangle \\ &= - \sum_{\ell(\lambda)=3, |\lambda|=0} \frac{1}{\lambda!} \left\langle \mathbf{a}_\lambda(\alpha) \sum_n c(T_{X^{[n]}}) q^n, |1\rangle \right\rangle \\ &= -\frac{1}{2} \left\langle (\mathbf{a}_{-1}\mathbf{a}_{-1}\mathbf{a}_2)(\alpha) \sum_n c(T_{X^{[n]}}) q^n, |1\rangle \right\rangle \\ &= -\frac{1}{2} \left\langle \mathbf{a}_2(\alpha) \sum_n c(T_{X^{[n]}}) q^n, |1\rangle \right\rangle. \end{aligned} \tag{6.13}$$

Set $\alpha = 1_X$. Put $\tilde{f}_\mu = \chi(X) \cdot f_\mu + \langle K_X, K_X \rangle \cdot h_\mu$. By Lemma 6.1, $F_1^{1_X}(q)$ equals

$$\begin{aligned} & -\frac{1}{2} \left\langle \mathbf{a}_2(1_X) \exp \left(\sum_{\mu \in \mathcal{P}} q^{|\mu|} \tilde{f}_\mu \mathbf{a}_{-\mu}(x) \right) |0\rangle, |1\rangle \right\rangle \\ &= -\frac{1}{2} \left\langle \mathbf{a}_2(1_X) \exp \left(\sum_{j \geq 0} q^{2+j} \tilde{f}_{(2,1^j)} \mathbf{a}_{-2}(x) \mathbf{a}_{-1}(x)^j \right) |0\rangle, |1\rangle \right\rangle \\ &= -\frac{1}{2} \left\langle \mathbf{a}_2(1_X) \left(\sum_{j \geq 0} q^{2+j} \tilde{f}_{(2,1^j)} \mathbf{a}_{-2}(x) \mathbf{a}_{-1}(x)^j \right) |0\rangle, |1\rangle \right\rangle \\ &= \sum_{j \geq 0} \tilde{f}_{(2,1^j)} q^{2+j}. \end{aligned}$$

(ii) Let $0 < |\alpha| < 4$. Again by (6.13) and Lemma 6.1, $F_1^\alpha(q)$ is equal to

$$\begin{aligned} & -\frac{1}{2} (q; q)_\infty^{-\chi(X)} \left\langle \mathbf{a}_2(\alpha) \exp \left(\sum_{\mu \in \mathcal{P}} q^{|\mu|} g_\mu \mathbf{a}_{-\mu}(K_X) \right) |0\rangle, |1\rangle \right\rangle \\ &= -\frac{1}{2} (q; q)_\infty^{-\chi(X)} \left\langle \mathbf{a}_2(\alpha) \exp \left(\sum_{j \geq 0} q^{2+j} g_{(2,1^j)} (\mathbf{a}_{-2} \mathbf{a}_{-1}^j)(K_X) \right) |0\rangle, |1\rangle \right\rangle \\ &= -\frac{1}{2} (q; q)_\infty^{-\chi(X)} \left\langle \mathbf{a}_2(\alpha) \left(\sum_{j \geq 0} q^{2+j} g_{(2,1^j)} \mathbf{a}_{-2}(K_X) \mathbf{a}_{-1}(x)^j \right) |0\rangle, |1\rangle \right\rangle. \end{aligned}$$

Therefore, $F_1^\alpha(q) = (q; q)_\infty^{-\chi(X)} \cdot \sum_{j \geq 0} g_{(2,1^j)} q^{2+j} \cdot \langle \alpha, K_X \rangle$ when $0 < |\alpha| < 4$. \square

Proposition 6.6. *Let the numbers $g_{(2,1^j)}$ and $h_{(2,1^j)}$ be from Lemma 6.1. Then, $g_{(2,1^j)} = -h_{(2,1^j)}$. Moreover, $\sum_{j \geq 0} g_{(2,1^j)} q^{2+j}$ is the coefficient of z^0 in*

$$\frac{1}{2} \left(\sum_n \frac{(n-1)q^n}{(1-q^n)^2} + \sum_n \frac{(qz)^n}{1-q^n} \cdot \left(\sum_m \frac{z^{-2m}}{(1-q^m)^2} + 2 \sum_{m_1 > m_2} \frac{z^{-m_1}}{1-q^{m_1}} \frac{z^{-m_2}}{1-q^{m_2}} \right) \right).$$

Proof. For simplicity, denote the previous line by $A(z)$. Let X be a smooth projective surface with $\chi(X) = 0$ and $\langle K_X, K_X \rangle \neq 0$. On one hand, applying Lemma 6.5 (i) and Proposition 4.13 to $F_1^{1_X}(q)$, we conclude that $\sum_{j \geq 0} h_{(2,1^j)} q^{2+j}$ is the coefficient of z^0 in $-A(z)$. On the other hand, applying Lemma 6.5 (ii) and Proposition 4.13 to $F_1^{K_X}(q)$, we see that $\sum_{j \geq 0} g_{(2,1^j)} q^{2+j}$ is the coefficient of z^0 in $A(z)$. It follows that $g_{(2,1^j)} = -h_{(2,1^j)}$ for every $j \geq 0$. \square

Remark 6.7. Let $N \geq 1$. Let $\alpha_1, \dots, \alpha_N \in H^*(X)$ be homogeneous classes such that $K_X \alpha_i = e_X \alpha_i = 0$ for all $1 \leq i \leq N$, and let $k_1, \dots, k_N \geq 0$.

(i) As in the proof of Lemma 6.3, we have

$$F_{k_1, \dots, k_N}^{\alpha_1, \dots, \alpha_N}(q) = (q; q)_\infty^{-\chi(X)} \left\langle \left(\prod_{i=1}^N \mathfrak{G}_{k_i}(\alpha_i) \right) \exp \left(\sum_{\mu \in \mathcal{P}} b_\mu \mathbf{a}_{-\mu}(1_X) q^{|\mu|} \right) |0\rangle, |1\rangle \right\rangle.$$

In principle, together with Theorem 4.8, this allows us to determine many of the universal constants b_μ in Lemma 6.1.

- (ii) In particular, $F_{k_1}^{\alpha_1}(q) = 0$ if $|\alpha_1| < 4$. This matches with Proposition 4.14 (i).

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